Game theory analysis for carbon auction market through electricity market coupling

Mireille Bossy, Odile Pourtallier, Nadia Maïzi

Design analysis mechanisms for carbon auction market through electricity market coupling

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Abstract In this paper, we analyze Nash equilibria between electricity producers selling their production on an electricity market and buying CO_2 emission allowances on an auction carbon market. The producers' strategies integrate the coupling of the two markets via the cost functions of the electricity production. We set out a clear Nash equilibrium on the power market that can be used to compute equilibrium prices on both markets as well as the related electricity produced and CO_2 emissions released.

1 Introduction

The aim of this paper is to develop analytic tools in order to design a relevant mechanism for carbon markets, where relevant refers to emissions reduction. For this purpose, we focus on electricity producers in a power market linked to a carbon market. The link between markets is established through a market microstructure approach. In this context, where the number of agents is limited, a standard game theory applies. The producers are considered as players behaving on the two financial markets represented here by carbon and electricity. We establish a Nash equilibrium for this non-cooperative J-player game through a coupling mechanism between the two markets.

The original idea comes from the French electricity sector, where the spot electricity market is often used to satisfy peak demand. Producers' behavior is demand

Mireille Bossy
Inria, France e-mail: mireille.bossy@inria.fr

Maïzi
MINES ParisTech, Centre for Applied Mathematics, CS 10207 rue Claude Daunesse 06904 Sophia Antipolis Cedex, France e-mail: nadia.maizi@mines-paristech.fr

Pourtallier
Inria, France e-mail: odile.pourtallier@inria.fr

driven and linked to the maximum level of electricity production. Each producer strives to maximize its market share. In the meantime, it has to manage the environmental burden associated with its electricity production through a mechanism inspired by the EU ETS[1] framework: each producer emission level must be counterbalanced by a permit or through the payment of a penalty. Emission permit allocations are simulated through a carbon market that allows the producers to buy allowances at an auction. Our focus on the electricity sector is motivated by its introduction in phase III of the EU ETS, and its prevalence in the emission share. In the present paper, the design assumptions made on the carbon market aim to foster emissions reduction in the entire electricity sector.

Based on a static elastic demand curve (referring to the time stages in an organized electricity market, mainly day-ahead and intra-day), we solve the local problem of establishing a non-cooperative Nash equilibrium for the two coupled markets.

While literature mainly addresses profit maximization, our share maximization approach deals with profit by making specific assumptions, i.e. no-loss sales, and a balance struck between the purchase of allowances and the carbon footprint of the electricity generated. Here the market is driven through demand dynamics rather than the electricity spot price dynamics used in recent works (see [5][4] [6]).

In Section 2, we formalize the market (carbon and electricity) rules and the associated admissible set of players' coupled strategies.

We start by studying (in section 3.2) the set of Nash equilibria on the electricity market alone (see Proposition 1). This set constitutes an equivalence class (same prices and market shares) from which we exhibit a dominant strategy.

Section 3.3 is devoted to the analysis of coupled markets equilibria: given a specific carbon market design (in terms of penalty level and allowances), we compute the bounds of the interval where carbon prices (derived from the previous dominant strategy) evolve. We specify the properties of the associated equilibria.

2 Coupling markets mechanism

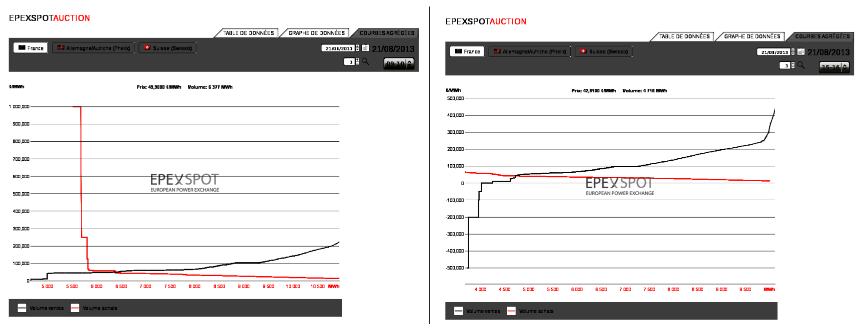
2.1 Electricity market

In the electricity market, demand is aggregated and summarized by a function $p \mapsto D(p)$, where $D(p)$ is the quantity of electricity that buyers are ready to obtain at maximal unit price p. We assume the following:

Assumption 1. *The demand function $D(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is decreasing, left continuous, and such that $D(0) > 0$.*

Each producer $j \in \{1, \ldots, J\}$ is characterized by a finite production capacity κ_j and a bounded and increasing function $c_j : [0, \kappa_j] \longrightarrow \mathbb{R}^+$ that associates a marginal production cost to any quantity q of electricity. These marginal production costs

[1] European Emission Trading System



(a) delivery 9-10 am (b) delivery 3-4 pm

Fig. 1 The orange curve is the function $q \mapsto D^{-1}(q)$ on the EPEX market. The evolution of the spot market confirms the relevance of Assumption 1 on the Demand function $p \mapsto D(p)$.

depend on several exogenous parameters reflecting the technical costs associated with electricity production e.g. energy prices, O&M costs, taxes, carbon penalties *etc*. This parameter dependency makes possible to build different market coupling mechanisms. In the following we use it to link the carbon and electricity markets.

The merit order ranking features marginal cost functions sorted according to their production costs. These are therefore increasing staircase functions whereby each stair refers to the marginal production cost of a specific unit owned by the producer.

The producers trade their electricity on a dedicated market. For a given producer j, the strategy consists in a function that makes it possible to establish an asking price on the electricity market, defined as

$$\begin{aligned} s_j : \mathscr{C}_j \times \mathbb{R}^+ &\longrightarrow \mathbb{R}^+ \\ (c_j(\cdot), q) &\longrightarrow s_j(c_j(\cdot), q), \end{aligned}$$

where $\mathscr{C}_j$ the set of marginal production cost functions **are explicitly given in the following** (see (13)).
$s_j(c_j(\cdot), q)$ is the unit price at which the producer is ready to sell quantity q of electricity. An admissible strategy carries out the following sell at no loss constraint

$$s_j(c_j(\cdot), q) \geq c_j(q), \quad \forall q \in \mathrm{Dom}(c_j). \tag{1}$$

For example we can take $s_j(c_j(\cdot), q) = c_j(q)$ or $s_j(c_j(\cdot), q) = c_j(q) + \lambda(q)$, where $\lambda(q)$ stands for any additional profit.
As mentioned in the introduction, the constraint (1) guarantees profitable trade as much as the equilibrium established through this class of strategy will benefit each producer. This establishes a link between market share maximization and profit maximization paradigms.

Let us denote $\mathbb{S}$ as the class of admissible strategy profiles on the electricity market. We have

$$\mathbb{S} = \left\{ \begin{array}{r} \mathbf{s} = (s_1, \ldots, s_J);\ s_j : \mathscr{C}_j \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \\ (c_j(\cdot), q) \longrightarrow s_j(c_j(\cdot), q) \\ \text{such that } s_j(c_j(\cdot), q) \geq c_j(q), \quad \forall q \in \mathrm{Dom}(c_j) \end{array} \right\}. \tag{2}$$

As a function of q, $s_j(c_j(\cdot),q)$ is bounded on $\mathrm{Dom}(c_j)$. For the sake of clarity, we define for each $q \notin \mathrm{Dom}(c_j)$, $s_j(c_j(\cdot),q) = p_{\mathrm{lolc}}$, where p_{lolc} is the loss of load cost, chosen as any overestimation of the maximal production costs.

For producer j's strategy s_j, we define the associated asking size at price p as

$$\mathscr{O}(c_j(\cdot), s_j; p) := \sup\{q,\ s_j(c_j(\cdot),q) < p\}. \tag{3}$$

Hence $\mathscr{O}(c_j(\cdot), s_j; p)$ is the maximum quantity of electricity at unit price p supplied by producer j on the market.

Remark 1.
(i) The asking size function $p \mapsto \mathscr{O}(c_j(\cdot), s_j; p)$ is, with respect to p, an increasing surjection from $[0,+\infty)$ to $[0,\kappa_j]$, right continuous and such that $\mathscr{O}(c_j(\cdot), s_j; 0) = 0$. For an increasing strategy s_j, $\mathscr{O}(c_j(\cdot), s_j; .)$ is its generalized inverse function with respect to q.
(ii) Given two strategies $q \mapsto s_j(c_j(\cdot),q)$ and $q \mapsto s'_j(c_j(\cdot),q)$ such that $s_j(c_j(\cdot),q) \leq s'_j(c_j(\cdot),q)$, for all $q \in Dom(c_j)$ we have for any positive p

$$\mathscr{O}(c_j(\cdot), s_j; p) \geq \mathscr{O}(c_j(\cdot), s'_j; p).$$

Indeed, if $p_1 \geq p_2$ then $\{q,\ s_j(c_j(\cdot),q) \leq p_2\} \subset \{q,\ s_j(c_j(\cdot),q) \leq p_1\}$ from which we deduce that $\mathscr{O}(c_j(\cdot), s_j; \cdot)$ is increasing. Next, if $s_j(c_j(\cdot),\cdot) \leq s'_j(c_j(\cdot),\cdot)$, for any fixed p, we have $\{q,\ s'_j(c_j(\cdot),q) \leq p\} \subset \{q,\ s_j(c_j(\cdot),q) \leq p\}$ from which the reverse order follows for the requests.

We shall now describe the electricity market clearing. Note that from a market view point, the dependency of the supply with respect to the marginal cost does not need to be explicit. For the sake of clarity, we write $s_j(q)$ and $\mathscr{O}(s_j; p)$ instead of $s_j(c_j(\cdot),q)$, $\mathscr{O}(c_j(\cdot), s_j; p)$. The dependency will be expressed explicitly whenever needed.

By aggregating the J asking size functions, we can define the overall asking function $p \mapsto \boldsymbol{\mathscr{O}}(\mathbf{s}; p)$ a producer strategy profile $\mathbf{s} = (s_1, \ldots, s_J)$ as:

$$\boldsymbol{\mathscr{O}}(\mathbf{s}; p) = \sum_{j=1}^{J} \mathscr{O}(s_j; p). \tag{4}$$

Hence, for any producer strategy profile $\mathbf{s}$, $\boldsymbol{\mathscr{O}}(\mathbf{s}; p)$ is the quantity of electricity that can be sold on the market at unit price p.

The overall supply function $p \mapsto \boldsymbol{\mathscr{O}}(\mathbf{s}; p)$ is an increasing surjection defined from $[0,+\infty)$ to $[0, \sum_{j=1}^{J} \kappa_j]$, such that $\boldsymbol{\mathscr{O}}(\mathbf{s}; 0) = 0$.

2.1.1 Electricity market clearing

Taking producer strategy profile $\mathbf{s} = (s_1(\cdot), \ldots, s_J(\cdot))$ the market sets the electricity market price $p^{\text{elec}}(\mathbf{s})$ together with the quantities $(\varphi_1(\mathbf{s}), \ldots, \varphi_J(\mathbf{s}))$ of electricity sold by each producer.

The market clearing price $p^{\text{elec}}(\mathbf{s})$ is the unit price paid to each producer for the quantities $\varphi_j(\mathbf{s})$ of electricity. The price $p(\mathbf{s})$ may be defined as a price whereby supply satisfies demand. As we are working with a general non-increasing demand curve (possibly locally inelastic), the price that satisfies the demand is not necessarily unique. We thus define the clearing price generically with the following definition.

Definition 1 (The clearing electricity price). Let us define

$$\begin{aligned} \underline{p}(\mathbf{s}) &= \inf\{p > 0;\ \mathscr{O}(\mathbf{s}; p) > D(p)\} \\ \text{and} \qquad & \\ \bar{p}(\mathbf{s}) &= \sup\{p \in [\underline{p}(\mathbf{s}), p_{\text{lolc}}]; D(p) = D(\underline{p}(\mathbf{s}))\} \end{aligned} \tag{5}$$

with the convention that $\inf \emptyset = p_{\text{lolc}}$. The clearing price may then be established as any $p^{\text{elec}}(\mathbf{s}) \in [\underline{p}(\mathbf{s}), \bar{p}(\mathbf{s})]$ as an output of a specific market clearing rule. To keep the price consistency, the market rule must be such that for any two strategy profiles $\mathbf{s}$ and $\mathbf{s}'$,

$$\begin{aligned} &\text{if } \underline{p}(\mathbf{s}) < \underline{p}(\mathbf{s}') \text{ then } p^{\text{elec}}(\mathbf{s}) < p^{\text{elec}}(\mathbf{s}'), \\ &\text{if } \underline{p}(\mathbf{s}) = \underline{p}(\mathbf{s}') \text{ then } p^{\text{elec}}(\mathbf{s}) = p^{\text{elec}}(\mathbf{s}'). \end{aligned} \tag{6}$$

Note that $\underline{p}(\mathbf{s}) \neq \bar{p}(\mathbf{s})$ only if the demand curve $p \mapsto D(p)$ is constant on some intervals $[\underline{p}(\mathbf{s}), \underline{p}(\mathbf{s}) + \varepsilon]$.

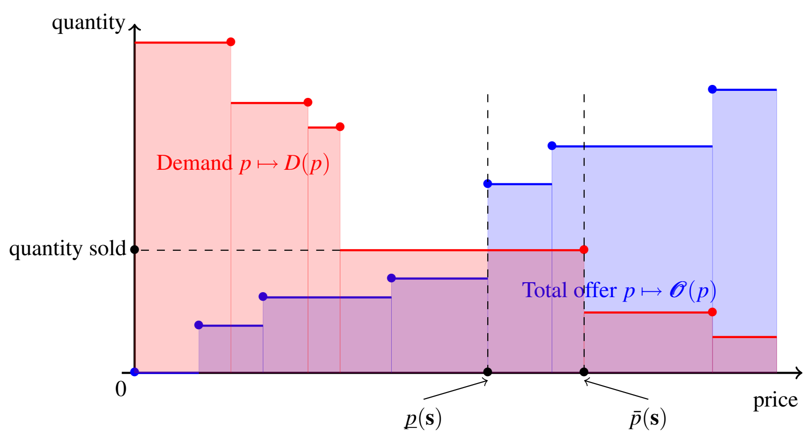


Fig. 2 Electricity clearing price $\underline{p}(\mathbf{s})$ and $\bar{p}(\mathbf{s})$.

Note also that price $\underline{p}(\mathbf{s})$ is well defined in the case where demand does not strictly decrease. This includes the case where demand is constant. In such case, $\underline{p}(\mathbf{s}) = p_{\text{lolc}}$ only if the demand curve never crosses the supply.

Next, we define the quantity of electricity sold at price $p^{\text{elec}}(\mathbf{s})$. When $\boldsymbol{\mathcal{O}}(\mathbf{s}; p^{\text{elec}}(\mathbf{s})) \leq D(p^{\text{elec}}(\mathbf{s}))$, each producer sells $\mathcal{O}(\mathbf{s}_j; p^{\text{elec}}(\mathbf{s}))$, but cases where $\boldsymbol{\mathcal{O}}(\mathbf{s}; p^{\text{elec}}(\mathbf{s})) > D(p^{\text{elec}}(\mathbf{s}))$ may occur, requiring the introduction of an auxiliary rule to share $D(p^{\text{elec}}(\mathbf{s}))$ among the producers that propose $\boldsymbol{\mathcal{O}}(\mathbf{s}; p^{\text{elec}}(\mathbf{s}))$. Note that in this last case, due to the clearing property (6) on $p^{\text{elec}}(\cdot)$, we have

$$\boldsymbol{\mathcal{O}}(\mathbf{s}; \underline{p}(\mathbf{s})) \geq D(p^{\text{elec}}(\mathbf{s})) = D(\underline{p}(\mathbf{s})).$$

Hence the $D(p^{\text{elec}}(\mathbf{s}))$ is totally provided by producers with non null offer at price $\underline{p}(\mathbf{s})$. The rule of the market is to share $D(p^{\text{elec}}(\mathbf{s}))$ among these producers only. This gives an explicit priority to the best offer prices $\underline{p}(\mathbf{s})$.

Let us break down supply as follows:

$$\boldsymbol{\mathcal{O}}(\mathbf{s}; \underline{p}(\mathbf{s})) = \sum_{j=1}^{J} \mathcal{O}(s_j; \underline{p}(\mathbf{s})^-) + \sum_{j=1}^{J} \Delta^- \mathcal{O}(s_j; \underline{p}(\mathbf{s})),$$

where $\Delta^- \mathcal{O}(s_j; \underline{p}(\mathbf{s})) := \mathcal{O}(s_j; \underline{p}(\mathbf{s})) - \mathcal{O}(s_j; \underline{p}(\mathbf{s})^-)$.

The market's choice is to fully accept the asking size of producers with continuous asking size curve at point $\underline{p}(\mathbf{s})$. For producers with discontinuous asking size curve at $\underline{p}(\mathbf{s})$, a market rule based on proportionality that favors abundance is used to share the remaining part of the supply. We resume the market rule on quantities as follows.

Definition 2 (Clearing electricity quantities). The quantity $\varphi_j(\mathbf{s})$ of electricity sold by Producer j on the electricity market is

$$\varphi_j(\mathbf{s}) = \begin{cases} \mathcal{O}(\mathbf{s}_j; p^{\text{elec}}(\mathbf{s})), \\ \quad \text{if } D(p^{\text{elec}}(\mathbf{s})) \geq \boldsymbol{\mathcal{O}}(\mathbf{s}; p^{\text{elec}}(\mathbf{s})), \\ \\ \mathcal{O}(\mathbf{s}_j; \underline{p}(\mathbf{s})^-) + \Delta^- \mathcal{O}(\mathbf{s}_j; \underline{p}(\mathbf{s})) \dfrac{D(\underline{p}(\mathbf{s})) - \boldsymbol{\mathcal{O}}(\mathbf{s}; \underline{p}(\mathbf{s})^-)}{\Delta^- \boldsymbol{\mathcal{O}}(\mathbf{s}; \underline{p}(\mathbf{s}))}, \\ \quad \text{if } D(p^{\text{elec}}(\mathbf{s})) < \boldsymbol{\mathcal{O}}(\mathbf{s}; \underline{p}(\mathbf{s})), \end{cases} \tag{7}$$

where $\Delta^- \boldsymbol{\mathcal{O}}(\mathbf{s}; \underline{p}(\mathbf{s})) := \sum_{j=1}^{J} \Delta^- \mathcal{O}(s_j; \underline{p}(\mathbf{s})) > 0$.

Note that, when $D(\underline{p}(\mathbf{s})) < \boldsymbol{\mathcal{O}}(\mathbf{s}; \underline{p}(\mathbf{s}))$, we have $\Delta^- \boldsymbol{\mathcal{O}}(\mathbf{s}; \underline{p}(\mathbf{s})) > 0$. Note also that we always have

$$\sum_{i=1}^{J} \varphi_j(\mathbf{s}) = D(p^{\text{elec}}(\mathbf{s})) \wedge \boldsymbol{\mathcal{O}}(\mathbf{s}; p^{\text{elec}}(\mathbf{s})). \tag{8}$$

2.2 Carbon market

Let us recall the CO_2 regulation principle on which we base our analysis. Producers are penalized according to their emission level if they do not own allowances. Hence, in parallel to their position on the electricity market, producers buy CO_2 emission allowances on a separate CO_2 auction market.

In the following, we formalize producer strategy on the CO_2 market only.

Assumption 2 (Capped carbon market).
(i) The carbon market is capped and has a finite known quantity Ω of CO_2 emission allowances available.
(ii) Each producer j can buy a capped number of allowances $\mathscr{E}_j$, related to its own CO_2 emission capacity.
(iii) Emissions that are not covered by allowances are penalized at a unit rate $\mathfrak{p}$.

On this market, producers adopt a strategy that consists in an offer function $\tau \mapsto A_j(\tau)$ defined from $[0,\mathfrak{p}]$ to $[0,\mathscr{E}_j]$. Quantity $A_j(\tau)$ is the quantity of allowances that producer j is ready to buy at price τ. This offer may not be a monotonic function. We denote $\mathbb{A}$ the strategy profile set on the CO_2 market,

$$\mathbb{A} := \{\mathbf{A} = (A_1,\dots,A_J);\ \text{s.t. } A_k : [0,\mathfrak{p}] \to [0,\mathscr{E}_j]\}.$$

The CO_2 market reacts by aggregating the J offers by

$$\mathscr{A}(\tau) := \sum_{j=1}^{J} A_j(\tau),$$

and the clearing market price is established following a *second item auction*[2] as:

$$p^{\text{CO}_2}(\mathbf{A}) := \sup\{\tau; \mathscr{A}(\tau) > \Omega\}, \qquad \text{with the convention } \sup\emptyset = 0. \tag{9}$$

Note that $p^{\text{CO}_2}(\mathbf{A}) = 0$ indicates that there are too many allowances to sell. It is worth a reminder here that the aim of allowances is to decrease emissions. In section 3.3, we discuss a design hypothesis (Assumption 6) that guarantees an equilibrium price $p^{\text{CO}_2}(\mathbf{A}) > 0$. Therefore, in the following, we assume that the overall quantity Ω of allowances, is such that $p^{\text{CO}_2}(\mathbf{A}) > 0$.

Next, we define the amount of allowances bought at price $p^{\text{CO}_2}(\mathbf{A})$ by the producers. By Definition (9), we have $\mathscr{A}(p^{\text{CO}_2}(\mathbf{A})) \geq \Omega$ and $\mathscr{A}(p^{\text{CO}_2}(\mathbf{A})^+) \leq \Omega$. When $\mathscr{A}(p^{\text{CO}_2}(\mathbf{A})) > \Omega$, the CO_2 market must decide between the producers with an additional rule. We define

$$\Delta(A_i) := A_i(p^{\text{CO}_2}(\mathbf{A})^+) - A_i(p^{\text{CO}_2}(\mathbf{A})).$$

[2] Also called *Dutch auction market* with several units to sell, in a *second item auction* market, the seller begins with a very high price and reduces it. The price is lowered until a bidder accepts the current price.

For a producer i, $\Delta(A_i) \geq 0$ means that its CO_2 demand does not decrease if the price increases. It is therefore ready to pay more to obtain the quantity of allowances it is asking for at price $p^{\mathrm{CO_2}}(\mathbf{A})$. The CO_2 market gives priority to this kind of producer, which will be fully served. The producers such that $\Delta(A_j) < 0$ share the remaining allowances. This can be written as follows.

Each producers with $A_j(p^{\mathrm{CO_2}}(\mathbf{A})) > 0$ obtains the following quantity $\delta_j(\mathbf{A})$ of allowances

$$\delta_j(\mathbf{A}) := \begin{cases} A_j(p^{\mathrm{CO_2}}(\mathbf{A})), & \text{if } \Delta(A_j) \geq 0, \\ A_j(p^{\mathrm{CO_2}}(\mathbf{A})^+) + \dfrac{(-\Delta(A_j))^+}{\sum_{i=1}^{J}(-\Delta(A_i))^+}\left(\Omega - \sum_{i=1}^{J} A_i(p^{\mathrm{CO_2}}(\mathbf{A}))\mathbb{1}_{\{\Delta(A_i)\geq 0\}}\right), \\ \quad \text{otherwise.} \end{cases} \tag{10}$$

2.3 Carbon and electricity market coupling

In the following, we formalize the coordination of a producer's strategy on the CO_2 and electricity markets.

As mentioned earlier, for each producer, the marginal cost function is parametrized by the positions $\mathbf{A}$ of the producers on the carbon market. Indeed, producer j can obtain CO_2 emission allowances on the market to avoid penalization for (some of) its emissions. Those emissions that are not covered by allowances are penalized at a unit rate $\mathfrak{p}$.

A profile of an offer to buy from the producers $\mathbf{A} = (A_1, \dots, A_J)$, through the CO_2 market clearing, corresponds to a unit price of $p^{\mathrm{CO_2}}(\mathbf{A})$ of the allowance and quantities $\delta_j(\mathbf{A})$ of allowances bought by each producer (defined by the market rules (9),(10)).

The following minimal assumption on the CO_2 emission related to the electricity production will be restricted in Assumption 4.

Assumption 3. *We assume that for all producers* $\{j = 1, \dots, J\}$*, the emission rate (originally in* CO_2 *t/Mwh)* $q \mapsto e_j(q)$ *is positive.*

For each producer, we fix the maximal amount $\mathscr{E}_j$ *of allowances to buy to* $\int_0^{\kappa_j} e_j(z)dz$*.*

Then, the marginal production cost function $c_j^{\mathbf{A}}(\cdot)$, parametrized by the emission regulations, comes out as

$$q \mapsto c_j^{\mathbf{A}}(q) = \begin{cases} c_j(q) + e_j(q)p^{\mathrm{CO_2}}(\mathbf{A}), & \text{for } q \in [0, \kappa_j^{\mathrm{CO_2}} \wedge \kappa_j] \\ c_j(q) + e_j(q)\mathfrak{p}, & \text{for } q \in [\kappa_j^{\mathrm{CO_2}} \wedge \kappa_j, \kappa_j] \end{cases} \tag{11}$$

where $\kappa_j^{\mathrm{CO_2}}$ is such that

$$\int_0^{\kappa_j^{\mathrm{CO}_2}} e_j(z)dz = \delta_j(\mathbf{A})$$

and where $c_j(\cdot)$ stands for the marginal production cost without any emission regulation.

Remark 2. *The results stated in Section 3.3 remain valid when the CO_2 regulation forbids uncovered electricity production. This strengthened regulation rule leads to the following marginal production cost function*

$$q \mapsto c_j^{\mathbf{A}}(q) = c_j(q) + e_j p^{CO_2}(\mathbf{A}),\ \textit{for } q \in [0, \kappa_j^{CO_2} \wedge \kappa_j].$$

In this coupled market setting, the strategy of producer j thus makes a pair (A_j, s_j). The set of admissible strategy profile is defined as

$$\boldsymbol{\Sigma} = \{(\mathbf{A}, \mathbf{s});\ \mathbf{A} \in \mathbb{A},\ \mathbf{s} \in \mathbb{S}\}, \tag{12}$$

where in the definition of $\mathbb{S}$ in (2), we use

$$\mathscr{C}_j = \left\{ c_j^{\mathbf{A}};\ \mathbf{A} \in \mathbb{A} \right\}. \tag{13}$$

Prices for allowances and electricity, $p^{\mathrm{CO}_2}((\mathbf{A}, \mathbf{s}))$ and $p^{\mathrm{elec}}((\mathbf{A}, \mathbf{s}))$, quantities of allowances bought by each producer, $\delta_j((\mathbf{A}, \mathbf{s}))$ and market shares on electricity market $\varphi_j((\mathbf{A}, \mathbf{s}))$ of each producer corresponds to any strategy profile $(\mathbf{A}, \mathbf{s}) \in \boldsymbol{\Sigma}$, through the market mechanisms described.

3 Nash Equilibrium analysis

3.1 Definition

We suppose that the J producers behave non cooperatively, aiming at maximizing their individual market share on the electricity market. For a strategy profile $(\mathbf{A}, \mathbf{s}) \in \boldsymbol{\Sigma}$, the market share of a producer j depends upon its strategy $(A_j, s_j(\cdot))$ but also on the strategies $(\mathbf{A}_{-j}, \mathbf{s}_{-j})$ of the other producers [3]. In this set-up the natural solution is the Nash equilibrium (see e.g. [1]). More precisely we are looking for a strategy profile

$$(\mathbf{A}^*, \mathbf{s}^*) = ((A_1^*, s_1^*), \cdots, (A_J^*, s_J^*)) \in \boldsymbol{\Sigma}$$

that satisfies Nash equilibrium conditions: none of the producers would strictly benefit, that is, would strictly increase its market share from a unilateral deviation. Namely, for any producer j strategy $(\mathbf{A}_j, \mathbf{s}_j)$ such that $((\mathbf{A}_{-j}^*, \mathbf{s}_{-j}^*); (A_j, s_j)) \in \boldsymbol{\Sigma}$, we have [4]

[3] Here $\mathbf{v}_{-j}$ stands for the profile $(v_i, \cdots, v_{j-1}, v_{j+1}, \cdots, v_J)$.

[4] $(\mathbf{v}_{-j}; v)$ stands for $(v_1, \cdots v_{j-1}, v, v_{j+1}, \cdots v_J)$

$$\varphi_j((\mathbf{A}^*,\mathbf{s}^*)) \geq \varphi_j((\mathbf{A}^*_{-j},\mathbf{s}^*_{-j});(A_j,s_j)), \tag{14}$$

where φ_j is the quantity of electricity sold. Note that the dependency in terms of $\mathbf{A}$ through the marginal cost $c_j^{\mathbf{A}}$ is now explicit in φ_j.

Condition (14) has to be satisfied for any unilateral deviation of any producer j. In particular (14) has to be satisfied for a producer j admissible deviation (A^*_j,s_j) such that $((\mathbf{A}^*_{-j},\mathbf{s}^*_{-j});(A^*_j,s_j)) \in \mathbf{\Sigma}$ where producer j would only change its behavior on the electricity market. Consequently,

Remark 3. *The electricity strategy component* $\mathbf{s}^*$ *of the Nash equilibrium* $(\mathbf{A}^*,\mathbf{s}^*)$ *is also a Nash equilibrium for the restricted electricity game, where producers only behave on the electricity market with marginal electricity production costs* $c_j^{\mathbf{A}^*}(\cdot)$, $j=1,\cdots J$.

The next section focuses on determining a Nash equilibrium on the game restricted to the electricity market.

3.2 Equilibrium on the power market

In this restricted set-up, we consider that the marginal costs $\{c_j, j=1\ldots,J\}$ are known data, possibly fixed through the position $\mathbf{A}$ on the CO_2 market. In this section, we refer to $\mathbb{S}$ as the set of admissible strategy profiles, in the particular case where $\mathscr{C}_j=\{c_j\}$ for each $j=1,\ldots,J$.

The Nash equilibrium problem is as follows: find a strategy profile $\mathbf{s}^*=(s^*_1,\ldots,s^*_J)\in\mathbb{S}$ such that

$$\forall j, \forall\, s_j \neq s^*_j, \quad \varphi_j(\mathbf{s}^*) \geq \varphi_j(\mathbf{s}^*_{-j};s_j). \tag{15}$$

The following proposition exhibits a Nash equilibrium, whereby each producer must choose the strategy denoted by C_j, and referred to as *marginal production cost strategy*. It is defined by

$$C_j(q) = \begin{cases} c_j(q), \text{ for } q \in \mathrm{Dom}(c_j) \\ p_{\mathrm{lolc}}, \text{ for } q \notin \mathrm{Dom}(c_j). \end{cases} \tag{16}$$

Proposition 1.
(i) For any strategy profile $\mathbf{s}=(s_1,\ldots,s_J)$, *no producer* $j\in\{1,\ldots,J\}$ *can be penalized by deviating from strategy* s_j *to its marginal production cost strategy* C_j, *namely,*

$$\varphi_j(\mathbf{s}) \leq \varphi(\mathbf{s}_{-j};C_j). \tag{17}$$

In other words, for any producer j, C_j *is a dominant strategy.*

(ii) The strategy profile $\mathbf{C} = (C_1, \dots C_J)$ *is a Nash equilibrium.*
(iii) If the strategy profile $\mathbf{s} \in \mathbb{S}$ *is a Nash equilibrium, then we have* $p^{elec}(\mathbf{s}) = p^{elec}(\mathbf{C})$ *and for any producer j,* $\varphi_j(\mathbf{s}) = \varphi_j(\mathbf{C})$.

Point (ii) of the previous proposition is a direct consequence of the dominance property (i). The proof of both (i) and (iii) can be found in [3]. Point (ii) of the proposition exhibits a Nash equilibrium strategy profile. Clearly this equilibrium is not unique since we can easily show that a producer's given supply can follow from countless different strategies. Nevertheless point (iii) shows that for any Nash equilibrium, the associated electricity prices are the same and the quantity of electricity bought by any producer j is the same for all equilibrium profiles.

Proof. We prove the dominance property *(i)*. Suppose that one producer, let us say producer 1, deviates and chooses C_1 instead of s_1. We have to show that its market share cannot be reduced by this deviation. By definition of the admissibility (see Equation (2) we have

$$s_1(q) \geq C_1(q), \forall q \in [0, \kappa_1].$$

Hence the offer functions defined by (3) satisfy

$$\mathscr{O}(s_1; \cdot) \leq \mathscr{O}(C_1; \cdot).$$

And by adding the unchanged offers of the other producers

$$\boldsymbol{\mathscr{O}}((\mathbf{s}_{-1}, s_1); \cdot) \leq \boldsymbol{\mathscr{O}}((\mathbf{s}_{-1}, C_1); \cdot), \tag{18}$$

where $(\mathbf{s}_{-1}; C_1)$ denotes the strategy profile that includes producer 1 deviation. The minimum market clearing price (5) for strategy profile $\mathbf{s}$ is :

$$\underline{p}(\mathbf{s}) = \inf\{p,\ \boldsymbol{\mathscr{O}}(\mathbf{s}; p) > D(p)\}.$$

The minimum market clearing price (5) for strategy profile $(\mathbf{s}_{-1}, C_1)$ is :

$$\underline{p}((\mathbf{s}_{-1}; C_1)) = \inf\{p,\ \boldsymbol{\mathscr{O}}((\mathbf{s}_{-1}; C_1); p) > D(p)\}$$

The inequality (18) together with the fact that the demand $D(\cdot)$ is a decreasing function imply that

$$\underline{p}((\mathbf{s}_{-1}; C_1)) \leq \underline{p}(\mathbf{s}).$$

From which, together with (6) we deduce that

$$p^{\text{elec}}((\mathbf{s}_{-1}; C_1)) \leq p^{\text{elec}}(\mathbf{s}).$$

Now let us show that producer 1 does not reduce its market share by deviating from $s_1(\cdot)$ to $C_1(\cdot)$, that is that

$$\varphi_1(\mathbf{s}_{-1}, C_1) \geq \varphi_1(\mathbf{s}).$$

For the sake of clarity we adopt, in this paragraph, the following notation:

$$\begin{array}{ll} \underline{p}_{\mathbf{s}} & := \underline{p}(\mathbf{s}) \\ p^{\text{elec}}_{\mathbf{s}} & := p^{\text{elec}}(\mathbf{s}) \\ \bar{p}_{\mathbf{s}} & := \bar{p}(\mathbf{s}) \end{array} \quad \text{and} \quad \begin{array}{ll} \underline{p}_{\mathbf{s}C} & := \underline{p}((\mathbf{s}_{-1};C_1)) \\ p^{\text{elec}}_{\mathbf{s}C} & := p^{\text{elec}}((\mathbf{s}_{-1};C_1)) \\ \bar{p}_{\mathbf{s}C} & := \bar{p}((\mathbf{s}_{-1};C_1)) \end{array}$$

We first consider the case where $p^{\text{elec}}_{\mathbf{s}C} < p^{\text{elec}}_{\mathbf{s}}$.

By definition of the minimum clearing price $\underline{p}_{\mathbf{s}C}$, the fact that $D(\underline{p}_{\mathbf{s}}) \leq D(\underline{p}_{\mathbf{s}C})$ and the fact that $\mathscr{O}((\mathbf{s}_{-1};C_1);\cdot)$ is non-decreasing, we have

$$D(\underline{p}_{\mathbf{s}}) \leq D(\underline{p}_{\mathbf{s}C}) \leq \mathscr{O}((\mathbf{s}_{-1};C_1);\underline{p}_{\mathbf{s}C}) \leq \mathscr{O}((\mathbf{s}_{-1};C_1);p^{\text{elec}}_{\mathbf{s}C}) \leq \mathscr{O}((\mathbf{s}_{-1};C_1);\bar{p}_{\mathbf{s}C})$$

Hence, for any $\pi_{\mathbf{s}} \in \{\underline{p}_{\mathbf{s}}, p^{\text{elec}}_{\mathbf{s}}, \bar{p}_{\mathbf{s}}\}$ and any $\pi_{\mathbf{s}C} \in \{\underline{p}_{\mathbf{s}C}, p^{\text{elec}}_{\mathbf{s}C}, \bar{p}_{\mathbf{s}C}\}$ we have

$$D(\pi_{\mathbf{s}}) \leq \mathscr{O}((\mathbf{s}_{-1};C_1);\pi_{\mathbf{s}C}).$$

Since $D(\pi_{\mathbf{s}}) \leq D(\pi_{\mathbf{s}C})$, we have

$$D(\pi_{\mathbf{s}}) \leq \mathscr{O}((\mathbf{s}_{-1};C_1);\pi_{\mathbf{s}C}) \wedge D(\pi_{\mathbf{s}C}),$$

and finally

$$\mathscr{O}((\mathbf{s}_{-1};s_1),\pi_{\mathbf{s}}) \wedge D(\pi_{\mathbf{s}}) \leq \mathscr{O}((\mathbf{s}_{-1};C_1);\pi_{\mathbf{s}C}) \wedge D(\pi_{\mathbf{s}C}).$$

From the market clearing we get

$$\begin{aligned} \varphi_1(\mathbf{s}_{-1},s_1) - \varphi_1(\mathbf{s}_{-1},C_1) =& \mathscr{O}((\mathbf{s}_{-1};s_1),p^{\text{elec}}_{\mathbf{s}}) \wedge D(p^{\text{elec}}_{\mathbf{s}}) - \mathscr{O}((\mathbf{s}_{-1};C_1);p^{\text{elec}}_{\mathbf{s}C}) \wedge D(p^{\text{elec}}_{\mathbf{s}C}) \\ &+ \sum_{j>1} \left(\varphi_j(\mathbf{s}_{-1},C_1) - \varphi_j(\mathbf{s}_{-1},s_1)\right). \end{aligned}$$

Let us denote

$$\mathscr{E}(p^{\text{elec}}_{\mathbf{s}}) = \left\{ j \in \{2,\ldots,J\} \text{ s.t. } \Delta^{-}\mathscr{O}(s_j;p^{\text{elec}}_{\mathbf{s}}) > 0 \right\}.$$

We have

$$\begin{aligned}\varphi_1(\mathbf{s}_{-1};s_1)-\varphi_1(\mathbf{s}_{-1};C_1)=&\boldsymbol{\mathscr{O}}((\mathbf{s}_{-1};s_1);p^{\text{elec}}_{\mathbf{s}})\wedge D(p^{\text{elec}}_{\mathbf{s}})-\boldsymbol{\mathscr{O}}((\mathbf{s}_{-1};C_1);p^{\text{elec}}_{\mathbf{s}C})\wedge D(p^{\text{elec}}_{\mathbf{s}C})\\&+\sum_{j>1,j\notin\mathscr{E}(p^{\text{elec}}_{\mathbf{s}})}\left(\varphi_j(\mathbf{s}_{-1};C_1)-\mathscr{O}(s_j;p^{\text{elec}}_{\mathbf{s}})\right)\\&+\sum_{j>1,j\in\mathscr{E}(p^{\text{elec}}_{\mathbf{s}})}(\varphi_j(\mathbf{s}_{-1};C_1)-\varphi_j(\mathbf{s}_{-1};s_1))\\\leq&\boldsymbol{\mathscr{O}}((\mathbf{s}_{-1};s_1);p^{\text{elec}}_{\mathbf{s}})\wedge D(p^{\text{elec}}_{\mathbf{s}})-\boldsymbol{\mathscr{O}}((\mathbf{s}_{-1};C_1);p^{\text{elec}}_{\mathbf{s}C})\wedge D(p^{\text{elec}}_{\mathbf{s}C})\\&+\sum_{j>1,j\notin\mathscr{E}(p^{\text{elec}}_{\mathbf{s}})}\left(\mathscr{O}(s_j;p^{\text{elec}}_{\mathbf{s}C})-\mathscr{O}(s_j;p^{\text{elec}}_{\mathbf{s}})\right)\\&+\sum_{j>1,j\in\mathscr{E}(p^{\text{elec}}_{\mathbf{s}})}(\varphi_j(\mathbf{s}_{-1};C_1)-\varphi_j(\mathbf{s}_{-1};s_1))\end{aligned}$$

Since $p^{\text{elec}}_{\mathbf{s}C}\leq p^{\text{elec}}_{\mathbf{s}}$ we get

$$\varphi_1(\mathbf{s}_{-1};s_1)-\varphi_1(\mathbf{s}_{-1};C_1)\leq\sum_{j>1\,j\in\mathscr{E}(p^{\text{elec}}_{\mathbf{s}})}(\varphi_j(\mathbf{s}_{-1};C_1)-\varphi_j(\mathbf{s}_{-1};s_1))$$

But for any $j\in\mathscr{E}(p^{\text{elec}}_{\mathbf{s}})$, the quantity $\mathscr{O}(s_j;\underline{p}_{\mathbf{s}}^-)\leq\varphi_j(\mathbf{s}_{-1};s_1)$. Hence since $\underline{p}_{\mathbf{s}C}<\underline{p}_{\mathbf{s}}$ and $\mathscr{O}(\mathbf{s}_j;\cdot)$ is non decreasing, we get

$$\mathscr{O}(s_j;\underline{p}_{\mathbf{s}C}^-)\leq\mathscr{O}(s_j;\underline{p}_{\mathbf{s}}^-)\leq\varphi_j(\mathbf{s}_{-1};s_1).$$

For such $j>1$ we thus have

$$\varphi_j(\mathbf{s}_{-1};C_1)-\varphi_j(\mathbf{s}_{-1};s_1)\leq\varphi_j(\mathbf{s}_{-1};C_1)-\mathscr{O}(s_j;p^{\text{elec}^-}_{\mathbf{s}})\leq\varphi_j(\mathbf{s}_{-1};C_1)-\mathscr{O}(s_j;p^{\text{elec}}_{\mathbf{s}C})\leq0.$$

from which it follows that

$$\varphi_1(\mathbf{s}_{-1};s_1)-\varphi_1(\mathbf{s}_{-1};C_1)\leq0.$$

Now consider the case where $p^{\text{elec}}_{\mathbf{s}}=p^{\text{elec}}_{\mathbf{s}C}:=p^{\text{elec}}$.

Due to the rule (6), we necessarily have $\underline{p}_{\mathbf{s}}=\underline{p}_{\mathbf{s}C}:=\underline{p}$.

• *If* $\boldsymbol{\mathscr{O}}((\mathbf{s}_{-1},s_1);p^{elec})\leq\boldsymbol{\mathscr{O}}((\mathbf{s}_{-1},C_1);p^{elec})\leq D(p^{elec})$,

hence by the market clearing

$$\varphi_1(\mathbf{s}_{-1};s_1)=\mathscr{O}(s_1;p^{\text{elec}})\leq\mathscr{O}(C_1;p^{\text{elec}})=\varphi_1(\mathbf{s}_{-1};C_1).$$

• *If* $\boldsymbol{\mathscr{O}}((\mathbf{s}_{-1};s_1);p^{elec})\leq D(p^{elec})\leq\boldsymbol{\mathscr{O}}((\mathbf{s}_{-1};C_1);p^{elec})$,

$$\begin{aligned}\varphi_1(\mathbf{s}_{-1};s_1) = \mathscr{O}(s_1;p^{\text{elec}}) \leq & D(p^{\text{elec}}) - \sum_{j>1} \varphi_j(\mathbf{s}_{-1};s_1) \\ &= D(p^{\text{elec}}) - \sum_{j>1} \mathscr{O}(s_j;p^{\text{elec}}) \\ &\leq D(p^{\text{elec}}) - \sum_{j>1} \varphi_j(\mathbf{s}_{-1};C_1) = \varphi_1(\mathbf{s}_{-1};C_1).\end{aligned}$$

• *If* $D(p^{elec}) \leq \boldsymbol{\mathscr{O}}((\mathbf{s}_{-1},s_1);p^{elec}) \leq \boldsymbol{\mathscr{O}}((\mathbf{s}_{-1},C_1);p^{elec})$,

by the market clearing we get

$$\begin{aligned}\varphi_1(\mathbf{s}_{-1};s_1) - \varphi_1(\mathbf{s}_{-1};C_1) =& \boldsymbol{\mathscr{O}}((\mathbf{s}_{-1};s_1),p^{\text{elec}}) \wedge D(p^{\text{elec}}) - \boldsymbol{\mathscr{O}}((\mathbf{s}_{-1};C_1);p^{\text{elec}}) \wedge D(p^{\text{elec}}) \\ &+ \sum_{j>1} (\varphi_j(\mathbf{s}_{-1};C_1) - \varphi_j(\mathbf{s}_{-1};s_1)) \\ \leq& \sum_{j>1} (\varphi_j(\mathbf{s}_{-1};C_1) - \varphi_j(\mathbf{s}_{-1};s_1)) \\ \leq& \sum_{j>1, j\in\mathscr{E}(p^{\text{elec}})} (\varphi_j(\mathbf{s}_{-1};C_1) - \varphi_j(\mathbf{s}_{-1};s_1)).\end{aligned}$$

From (7), we have for $j \in \mathscr{E}(\bar{p})$

$$\varphi_j(\mathbf{s}_{-1};s_1) = \mathscr{O}(s_j,\underline{p}^-) + \Delta^-\mathscr{O}(s_j;\bar{p}) \frac{(D(\bar{p}) - \boldsymbol{\mathscr{O}}((\mathbf{s}_{-1};s_1),\underline{p}^-))}{\Delta^-\boldsymbol{\mathscr{O}}((\mathbf{s}_{-1};s_1),\bar{p})}$$

and

$$\varphi_j(\mathbf{s}_{-1};C_1) = \mathscr{O}(s_j;\underline{p}^-) + \Delta^-\mathscr{O}(s_j;\bar{p}) \frac{(D(\bar{p}) - \boldsymbol{\mathscr{O}}((\mathbf{s}_{-1};C_1);\underline{p}^-))}{\Delta^-\boldsymbol{\mathscr{O}}((\mathbf{s}_{-1};C_1);\bar{p})}.$$

Hence, if $\mathscr{E}(\bar{p})$ is non empty then at least one producer exists, $j \neq 1$ such that $\Delta^-\mathscr{O}(s_j;\bar{p}) > 0$. and from the desegregation of $\boldsymbol{\mathscr{O}}$ and definition of Δ^- it results that

$$\begin{aligned}&\varphi_1(\mathbf{s}_{-1};s_1) - \varphi_1(\mathbf{s}_{-1};C_1) \\ &= \sum_{j>1, j\in\mathscr{E}(\bar{p})} \Delta^-\mathscr{O}(s_j,\bar{p}) \Bigg(\frac{(D(\bar{p}) - \boldsymbol{\mathscr{O}}(\mathbf{s}_{-1};\underline{p}^-) - \mathscr{O}(C_1;\underline{p}^-))}{\boldsymbol{\mathscr{O}}((\mathbf{s}_{-1};C_1);\bar{p}) - \boldsymbol{\mathscr{O}}(\mathbf{s}_{-1};\underline{p}^-) - \mathscr{O}(C_1;\underline{p}^-)} \\ &\qquad - \frac{(D(\bar{p}) - \boldsymbol{\mathscr{O}}(\mathbf{s}_{-1},\underline{p}^-) - \mathscr{O}(s_1,\underline{p}^-))}{\boldsymbol{\mathscr{O}}((\mathbf{s}_{-1},s_1),\bar{p}) - \boldsymbol{\mathscr{O}}(\mathbf{s}_{-1};\underline{p}^-) - \mathscr{O}(s_1;\underline{p}^-)} \Bigg)\end{aligned}$$

We note that

$$\begin{aligned}0 < \boldsymbol{\mathscr{O}}((\mathbf{s}_{-1};s_1);\bar{p}) - \boldsymbol{\mathscr{O}}(\mathbf{s}_{-1};\underline{p}^-) - \mathscr{O}(C_1;\underline{p}^-) \\ \leq \boldsymbol{\mathscr{O}}((\mathbf{s}_{-1};C_1);\bar{p}) - \boldsymbol{\mathscr{O}}(\mathbf{s}_{-1};\underline{p}^-) - \mathscr{O}(C_1;\underline{p}^-)\end{aligned}$$

and that $D(\bar{p}) - \boldsymbol{\mathscr{O}}((\mathbf{s}_{-1};C_1);\underline{p}^-) > 0$ by assumption. Then

$$\varphi_1(\mathbf{s}_{-1};s_1)-\varphi_1(\mathbf{s}_{-1};C_1)$$
$$\leq \sum_{j>1, j\in\mathscr{E}(\bar p)} \Delta^-\mathscr{O}(s_j;\bar p)\times\Bigg(\frac{(D(\bar p)-\boldsymbol{\mathscr{O}}(\mathbf{s}_{-1};\bar p_-)-\mathscr{O}(C_1;\bar p_-))}{\boldsymbol{\mathscr{O}}((\mathbf{s}_{-1};s_1);\bar p)-\boldsymbol{\mathscr{O}}(\mathbf{s}_{-1};\underline{p}^-)-\mathscr{O}(C_1;\underline{p}^-)}$$
$$-\frac{(D(\bar p)-\boldsymbol{\mathscr{O}}(\mathbf{s}_{-1};\bar p_-)-\mathscr{O}(s_1;\bar p_-))}{\boldsymbol{\mathscr{O}}((\mathbf{s}_{-1};s_1);\bar p)-\boldsymbol{\mathscr{O}}(\mathbf{s}_{-1};\underline{p}^-)-\mathscr{O}(s_1;\underline{p}^-)}\Bigg)$$

Since $D(\bar p)\leq\boldsymbol{\mathscr{O}}((\mathbf{s}_{-1};s_1);\bar p)$ and $\mathscr{O}(C_1;\underline{p}^-)\geq\mathscr{O}(s_1;\underline{p}^-)$, we can deduce that

$$\varphi_1(\mathbf{s}_{-1};s_1)-\varphi_1(\mathbf{s}_{-1};C_1)\leq 0.$$

This follows from the fact that $x\mapsto\dfrac{A-x}{B-x}$ with $A\leq B$, is decreasing with respect to x. □

3.3 *Coupled market design using the Nash equilibrium*

From this point we restrict our attention to a particular market design. In the following, the scope of the analysis applies to a special class of producers, a specific electricity market price clearing (satisfying Definition 1) and a range of quantities Ω of allowances available on the CO_2 market. Although not necessary, the following restriction simplifies the development.

Assumption 4. On the producers. *Each producer j operates a single production unit, for which*

(i) The marginal cost contribution that does not depend on the producer positions $\mathbf{A}$ in the CO_2 market is constant, $q\mapsto c_j(q)=c_j$. The related emission rate $q\mapsto e_j(q)=e_j$ is also assumed to be a positive constant.
(ii) The producers are different pairwise: $\forall i,j\in\{1,\cdots J\},(c_i,e_i)\neq(c_j,e_j)$.

For each producer, the maximal amount $\mathscr{E}_j$ of allowances to buy is now $e_j\kappa_j$.

As a consequence of Assumption 4, the marginal production cost in (11) simply writes as

$$q\mapsto c_j^{\mathbf{A}}(q)=\begin{cases} c_j+e_jp^{\mathrm{CO_2}}(\mathbf{A}), & \text{for } q\in[0,\dfrac{\delta_j(\mathbf{A})}{e_j}\wedge\kappa_j] \\ c_j+e_j\mathfrak{p}, & \text{for } q\in[\dfrac{\delta_j(\mathbf{A})}{e_j}\wedge\kappa_j,\kappa_j]\end{cases}\tag{19}$$

For a given strategy profile on the electricity market, Definition 1 gives a range of possible determinations for the electricity price. Previously, the analysis of the Nash Equilibrium restricted to the electricity market did not require a precise clearing price determination. Nevertheless to extend our analysis of the coupling we need to explicit this determination and assume the following:

Assumption 5. On the electricity market. *For a given strategy profile* $\mathbf{s}$ *of the producers, the clearing price of electricity is* $p^{elec}(\mathbf{s})$*. The market rule fixes* $p^{elec}(\cdot) = \bar{p}(\cdot)$ *or* $p^{elec}(\cdot) = \underline{p}(\cdot)$ *as defined in* (5).

We will illustrate below that this choice of clearing price ensures the increasing behavior of $p^{\mathrm{elec}}(\cdot)$ in terms of the carbon price (see Lemma 1).

The quantity Ω of CO_2 allowances available on the market plays a crucial role in the market design. As a matter of fact, if this quantity is too high, its market price will drop to zero, leaving the market incapable of fulfilling its role of decreasing CO_2 emissions. Therefore we clearly need to make an assumption that restricts the number of allowances available. Capping the maximum quantity of allowances available requires information on which producers are willing to obtain allowances. This is the objective of the following paragraph where we define a *willing to buy* function that plays a central role in the analysis of Nash equilibria.

3.3.1 Willing to buy functions

In this paragraph, we aim at guessing a Nash equilibrium candidate. We base our reasoning on the dominant strategies on the electricity market alone (see Proposition 1). Remark 3 allows us to fix the electricity market strategy as a *marginal production cost strategy*, given the marginal cost functions $\mathbf{C^A} = \{c_j^{\mathbf{A}}, j = 1, \dots J\}$ imposed by the output of the CO_2 clearing, as in (19).

In particular, when $\mathbf{A} \in \mathbb{A}$, we observe that the strategies $(\mathbf{A}, \{c_j^{\mathbf{A}}, j = 1, \dots J\})$ are in the set of admissible strategies defined in (12).

From now on, all the strategy profiles that we consider on the carbon market are assumed to be admissible.

In the following, as the discussion will mainly focus on the impact of strategies $\mathbf{A}$ through the carbon market, we denote the electricity market output as:

$$\begin{array}{rcl} p^{\mathrm{elec}}(\mathbf{A}) & \text{instead of} & p^{\mathrm{elec}}(\mathbf{C^A}) \\ (\varphi_1(\mathbf{A}), \dots, \varphi_J(\mathbf{A})) & \text{instead of} & (\varphi_1(\mathbf{C^A}), \dots, \varphi_J(\mathbf{C^A})). \end{array} \tag{20}$$

To begin with, we consider an exogenous CO_2 cost τ similar to a CO_2 tax: the producers' marginal cost becomes for any $\tau \in [0, \mathfrak{p}]$, $c_j^{\tau}(\cdot)$,

$$c_j^{\tau}(q) = c_j + \tau e_j, \text{ for } q \in [0, \kappa_j], \ j = 1, \dots, J.$$

In this *tax* framework, the dominant strategy on the electricity market is also parametrized by τ as $\mathbf{C}^{\tau} = \{c_j^{\tau}, j = 1, \dots J\}$ defined in (16). The clearing electricity price and quantities derive as

$$\begin{aligned} p^{\mathrm{elec}}(\tau) &= p^{\mathrm{elec}}(\mathbf{C}^{\tau}) \\ (\varphi_1(\tau), \dots, \varphi_J(\tau)) &= (\varphi_1(\mathbf{C}^{\tau}), \dots, \varphi_J(\mathbf{C}^{\tau})). \end{aligned} \tag{21}$$

Price $p^{\text{elec}}(\tau)$ will be referred to as the *taxed* electricity price, by contrast with price $p^{\text{elec}}(\mathbf{A})$ issued from the *marginal production cost strategy* that results from the position $\mathbf{A}$ on the carbon market.

Remark 4. *Considering a carbon tax τ and a carbon market strategy $\mathbf{A}$ such that $\tau = p^{co_2}(\mathbf{A})$, we emphasize the fact that the corresponding electricity prices are not equivalent, but we always have the following inequality*

$$p^{elec}(\tau) \leq p^{elec}(\mathbf{A}).$$

This comes from the fact that $\mathbf{C}^\tau$ and $\mathbf{C}^A$ differ only on the width of their steps, and that $\mathscr{O}(c_i^{\mathbf{A}};\cdot) \leq \mathscr{O}(c_i^{\tau};\cdot)$.

We start with the following, the proof of which is set out at the end of this subsection:

Lemma 1. *Under Assumption 5, the map $\tau \mapsto p^{elec}(\tau)$ is increasing and right continuous.*

We determine the *willing-to-buy-allowances functions* $\mathscr{W}_j(\cdot)$ and $\mathscr{W}(\cdot)$, as follows:

$$\mathscr{W}_j(\tau) = e_j \varphi_j(\tau) \quad \text{and} \quad \mathscr{W}(\tau) = \sum_{j=1}^{J} \mathscr{W}_j(\tau) \tag{22}$$

For producer j, $\mathscr{W}_j$ is the quantity of emissions it would produce under the penalization τ, and consequently the quantity of allowances it would be ready to buy at price τ. Given the CO_2 value τ, the total amount $\mathscr{W}(\tau)$ represents the allowances needed to cover the global emissions generated by the players who have won electricity market shares. We also define the functions

$$\overline{\mathscr{W}}_j(\tau) = e_j \kappa_j \mathbb{1}_{\{\varphi_j(\tau)>0\}}, \quad \text{and} \quad \overline{\mathscr{W}}(\tau) = \sum_{j=1}^{J} \overline{\mathscr{W}}_j(\tau) \tag{23}$$

Given that the CO_2 value τ, $\overline{\mathscr{W}}(\tau)$ is the amount of allowances needed by the producers who have won electricity market shares and want to cover their overall production capacity κ_j. Obviously we have

$$\mathscr{W}(\tau) \leq \overline{\mathscr{W}}(\tau), \; \forall \tau \in [0, \mathfrak{p}].$$

We now can state our last design assumption

Assumption 6. On the carbon market design. *The available allowances Ω satisfy*

$$\overline{\mathscr{W}}(\mathfrak{p}) < \Omega < \mathscr{W}(0).$$

Assumption 6 allows us to define two prices of particular interest for the construction of the equilibrium strategy:

$$\tau^{\text{lower}} = \sup\{\tau \in [0,\mathfrak{p}] \text{ s.t. } \mathscr{W}(\tau) > \Omega\}, \tag{24}$$

and

$$\tau^{\text{higher}} = \sup\{\tau \in [0,\mathfrak{p}] \text{ s.t. } \overline{\mathscr{W}}(\tau) > \Omega\}. \tag{25}$$

Observe that we always have $\tau^{\text{lower}} \leq \tau^{\text{higher}}$.

Lemma 2. *The function $\tau \mapsto \mathscr{W}(\tau)$ is non increasing:*

$$\mathscr{W}(t') \leq \mathscr{W}(t),\ \forall\, 0 \leq t < t' \leq \mathfrak{p}.$$

We end this subsection by successively giving the proofs of Lemmas 1 and 2.

Proof of Lemma 1. Although the result of this lemma is intuitive, the proof is rather technical. This is due to our assumptions, in particular regarding demand, that allow the demand function to have discontinuity points and some non-elasticity areas (see Assumption 1).

More precisely, if we define the map $\tau \mapsto \boldsymbol{\mathscr{O}}(\tau;p)$ by

$$\boldsymbol{\mathscr{O}}(\tau;p) = \sum_{i=1}^{J} \mathscr{O}(C_j^{\tau}(\cdot);p) = \sum_{i=1}^{J} \kappa_i \mathbb{1}_{\{p \geq c_i + \tau e_i\}} = \sum_{i=1}^{J} \kappa_i \mathbb{1}_{\{\tau \leq \frac{p-c_i}{e_i}\}},$$

then we can observe that, for any $p > 0$ far enough from the c_i, for any $\varepsilon \geq 0$

$$\boldsymbol{\mathscr{O}}(\tau+\varepsilon;p) \leq \boldsymbol{\mathscr{O}}(\tau;p)$$

and

$$\lim_{\varepsilon \to 0, \varepsilon > 0} \boldsymbol{\mathscr{O}}(\tau+\varepsilon;p) = \boldsymbol{\mathscr{O}}(\tau;p).$$

We call $S_D = \{p_d; \lim_{\varepsilon \to 0; \varepsilon > 0} D(p_d+\varepsilon) < D(p_d)\}$, the set of discontinuity points of the Demand function.

We call $S_\kappa = \{p_c; D(p_c) = \sum \kappa_i\}$, the set of prices that make demand coincide with some accumulation of production capacities.

We observe that $p^{\text{elec}}(\tau) \in \{c_i + \tau e_i, i = 1,\ldots,j\} \cup S_D \cup S_\kappa$. In particular, from Definition 1,

$$\underline{p}(\tau) = \inf\{p > 0; \boldsymbol{\mathscr{O}}(\tau;p) > D(p)\},$$

and we obtain that

$$D(\underline{p}(\tau+\varepsilon)) \leq \boldsymbol{\mathscr{O}}(\tau+\varepsilon;\underline{p}(\tau+\varepsilon)) \leq \boldsymbol{\mathscr{O}}(\tau;\underline{p}(\tau+\varepsilon))$$

from which we conclude that $\underline{p}(\tau+\varepsilon) \geq \underline{p}(\tau)$.

Now we prove the right continuity of $\tau \mapsto \underline{p}(\tau)$. Let us fix a τ.

(i) We first consider the case $D(\underline{p}(\tau)) < \mathscr{O}(\tau;\underline{p}(\tau))$.

This means that $\underline{p}(\tau)$ is of the form $c_\ell + \tau\ell$, for a given ℓ. Then when $\varepsilon > 0$ is small enough, we also have $\underline{p}(\tau+\varepsilon) = c_\ell + (\tau+\varepsilon)e_\ell$. Indeed, $D(c_\ell + (\tau+\varepsilon)e_\ell) \leq D(c_\ell + \tau e_\ell)$ and for a small enough ε,

$$\mathscr{O}(\tau;c_\ell + \tau e_\ell) = \kappa_\ell + \sum_{i\neq\ell} \kappa_i \mathbb{1}_{\{\tau \leq \frac{c_\ell - c_i}{1-e_i/e_\ell}\}} = \mathscr{O}(\tau+\varepsilon;c_\ell + (\tau+\varepsilon)e_\ell)$$

Thus, $D(c_\ell + (\tau+\varepsilon)e_\ell) < \mathscr{O}(\tau+\varepsilon;c_\ell + (\tau+\varepsilon)e_\ell)$ which implies that $\underline{p}(\tau) + e_\ell\varepsilon = c_\ell + (\tau+\varepsilon)e_\ell \geq \underline{p}(\tau+\varepsilon)$ and hence

$$e_\ell\varepsilon \geq \underline{p}(\tau+\varepsilon) - \underline{p}(\tau).$$

(ii) We consider next the case $D(\underline{p}(\tau)) > \mathscr{O}(\tau;\underline{p}(\tau))$.

This means that $\underline{p}(\tau) \in S_D$ is at a discontinuity point, say p_d of the demand, $\underline{p}(\tau) = p_d$. Then, for any $\delta > 0$,

$$D(\underline{p}(\tau)+\delta) < \mathscr{O}(\tau;\underline{p}(\tau)+\delta).$$

But,

$$\mathscr{O}(\tau;p_d+\delta) = \sum_{i=1}^{J} \kappa_i \mathbb{1}_{\{\tau \leq \frac{p_d+\delta-c_i}{e_i}\}}$$

and we can choose δ to be small enough so that $\tau \neq \frac{p_d+\delta-c_i}{e_i}$. Then, for a small enough ε,

$$D(\underline{p}(\tau)+\delta) < \mathscr{O}(\tau;\underline{p}(\tau)+\delta) = \mathscr{O}(\tau+\varepsilon;\underline{p}(\tau)+\delta)$$

which implies that $\underline{p}(\tau)+\delta \geq \bar{p}(\tau+\varepsilon)$, so we obtain

$$\delta \geq \underline{p}(\tau+\varepsilon) - \underline{p}(\tau) \geq 0.$$

(iii) We consider now the case $D(\underline{p}(\tau)) = \mathscr{O}(\tau;\underline{p}(\tau))$.

This means that $\underline{p}(\tau) \in S_\kappa$, say $\underline{p}(\tau) = p_c$ Then, for any $\delta > 0$,

$$D(\underline{p}(\tau)+\delta) < \mathscr{O}(\tau;\underline{p}(\tau)+\delta).$$

But,

$$\mathscr{O}(\tau;p_c+\delta) = \sum_{i=1}^{J} \kappa_i \mathbb{1}_{\{\tau \leq \frac{p_c+\delta-c_i}{e_i}\}}$$

and we can choose δ small enough such that $\tau \neq \frac{p_c+\delta-c_i}{e_i}$. Then, for ε small enough,

$$D(\underline{p}(\tau)+\delta) < \mathscr{O}(\tau;\underline{p}(\tau)+\delta) = \mathscr{O}(\tau+\varepsilon;\underline{p}(\tau)+\delta)$$

which implies that $\underline{p}(\tau)+\delta \geq \underline{p}(\tau+\varepsilon)$, so we get

$$\delta \geq \underline{p}(\tau+\varepsilon) - \underline{p}(\tau) \geq 0.$$

The right-continuity of $\tau \mapsto \bar{p}(\tau)$ follows, by definition as $\bar{p}(\tau)$ is a continuous transformation of $\underline{p}(\tau)$. □

Proof of Lemma 2. The proof consists in a complete analysis of the entire combination of situations, but each situation is elementary. We present some cases here, but we relegate the rest in the appendix.

Let us suppose the opposite, that is there exists $0 \leq t < t' \leq \mathfrak{p}$ such that the emission levels are $\mathscr{W}(t') > \mathscr{W}(t)$.

We define the function $\tau \mapsto I(\tau)$ valued in the subsets of $\{1,\dots,J\}$ that lists the producers in the electricity market producing at tax level τ:

$$i \in I(\tau) \quad \text{if} \quad \varphi_i(\tau) > 0.$$

In particular we have for all $\tau \in [0,\mathfrak{p}]$,

$$\mathscr{W}(\tau) = \sum_{i\in I(\tau)} e_i \varphi_i(\tau).$$

(i) We first examine the situation $I(t') = I(t)$.

To shorten the expressions, we adopt the following shorten notation

$$I(t) = I \quad \text{and} \quad I(t') = I'.$$

(i-a) If $\sum_{i\in I}\varphi_i(t) = D(t)$ then, from the demand constraint (DC) and the emission levels hypothesis (EH), we have

$$\sum_{i\in I}\varphi_i(t) = D(t) \geq D(t') \geq \sum_{i\in I'}\varphi_i(t') \quad (DC)$$
$$\sum_{i\in I}\varphi_i(t)e_i < \sum_{i\in I'}\varphi_i(t')e_i. \quad (EH)$$

We denote by $\widehat{I}$ the subset of I of index such that $c_i + te_i = \underline{p}(t)$. In particular, when $j \in I\setminus\widehat{I}$, then $\varphi_j(t) = \kappa_j$.

Note that there exists at most one index (say ℓ) in the set $\widehat{I}\cap\widehat{I'}$. If $j \in \widehat{I}\setminus\widehat{I'}$, if $k \in \widehat{I'}\setminus\widehat{I}$, then, by the definition of the sets

$$\begin{aligned}
c_j + e_j t &= c_\ell + e_\ell t \\
c_j + e_j t' &< c_\ell + e_\ell t' \\
c_k + e_k t &< c_j + e_j t \\
c_k + e_k t' &= c_\ell + e_\ell t' \\
c_k + e_k t &< c_\ell + e_\ell t \\
c_j + e_j t' &< c_k + e_k t
\end{aligned}$$

From which, we easily deduce that

$$\max\{e_j, j \in \widehat{I} \setminus \widehat{I'}\} < e_\ell < \min\{e_k, k \in \widehat{I'} \setminus \widehat{I}\}. \tag{26}$$

Now, we decompose the sets I and I' in the demand constraint (DC) and the emission levels hypothesis (EH) as follows:

$$\begin{aligned}
&\sum_{n \in I \setminus \widehat{I} \cup \widehat{I'}} \kappa_n + \varphi_\ell(t) + \sum_{i \in \widehat{I} \setminus \widehat{I'}} \varphi_i(t) + \sum_{k \in \widehat{I'} \setminus \widehat{I}} \kappa_k \\
&\geq \sum_{n \in I \setminus \widehat{I} \cup \widehat{I'}} \kappa_n + \varphi_\ell(t') + \sum_{i \in \widehat{I} \setminus \widehat{I'}} \kappa_i + \sum_{k \in \widehat{I'} \setminus \widehat{I}} \varphi_k(t') \quad \text{(DC)}
\end{aligned}$$

$$\begin{aligned}
&\sum_{n \in I \setminus \widehat{I} \cup \widehat{I'}} e_n \kappa_n + e_\ell \varphi_\ell(t) + \sum_{i \in \widehat{I} \setminus \widehat{I'}} e_i \varphi_i(t) + \sum_{k \in \widehat{I'} \setminus \widehat{I}} e_k \kappa_k \\
&< \sum_{n \in I \setminus \widehat{I} \cup \widehat{I'}} e_n \kappa_n + e_\ell \varphi_\ell(t') + \sum_{i \in \widehat{I} \setminus \widehat{I'}} e_i \kappa_i + \sum_{k \in \widehat{I'} \setminus \widehat{I}} e_k \varphi_k(t') \quad \text{(EH)}\,.
\end{aligned}$$

After simplification, we obtain

$$\begin{aligned}
&\varphi_\ell(t) + \sum_{i \in \widehat{I} \setminus \widehat{I'}} \varphi_i(t) + \sum_{k \in \widehat{I'} \setminus \widehat{I}} \kappa_k \geq \varphi_\ell(t') + \sum_{i \in \widehat{I} \setminus \widehat{I'}} \kappa_i + \sum_{k \in \widehat{I'} \setminus \widehat{I}} \varphi_k(t') \quad \text{(DC)} \\
&e_\ell \varphi_\ell(t) + \sum_{i \in \widehat{I} \setminus \widehat{I'}} e_i \varphi_i(t) + \sum_{k \in \widehat{I'} \setminus \widehat{I}} e_k \kappa_k < e_\ell \varphi_\ell(t') + \sum_{i \in \widehat{I} \setminus \widehat{I'}} e_i \kappa_i + \sum_{k \in \widehat{I'} \setminus \widehat{I}} e_k \varphi_k(t') \quad \text{(EH)}\,.
\end{aligned}$$

Assume first that $\varphi_\ell(t) + \sum_{i \in \widehat{I} \setminus \widehat{I'}} \varphi_i(t) \geq \varphi_\ell(t') + \sum_{i \in \widehat{I} \setminus \widehat{I'}} \kappa_i$. Equivalently, we have

$$\varphi_\ell(t) - \varphi_\ell(t') \geq \sum_{i \in \widehat{I} \setminus \widehat{I'}} (\kappa_i - \varphi_i(t))$$

and from (26),

$$e_\ell \left(\varphi_\ell(t) - \varphi_\ell(t')\right) \geq \sum_{i \in \widehat{I} \setminus \widehat{I'}} e_i (\kappa_i - \varphi_i(t)).$$

By combining the above with the emission levels hypothesis (EH), we obtain the following contradiction: $\sum_{k \in \widehat{I'} \setminus \widehat{I}} e_k \kappa_k < \sum_{k \in \widehat{I'} \setminus \widehat{I}} e_k \varphi_k(t')$.

Assume now that $\varphi_\ell(t)+\sum_{i\in\widehat{I}\setminus\widehat{I'}}\varphi_i(t) < \varphi_\ell(t')+\sum_{i\in\widehat{I}\setminus\widehat{I'}}\kappa_i$. Multiplying the demand constraint (DC) by $\hat{e}:=\min\{e_k, k\in\widehat{I'}\setminus\widehat{I}\}$, we get

$$\sum_{k\in\widehat{I'}\setminus\widehat{I}} e_k(\kappa_k-\varphi_k(t'))\geq \hat{e}\left(\varphi_\ell(t)-\varphi_\ell(t')\right)+\hat{e}\sum_{i\in\widehat{I}\setminus\widehat{I'}}(\kappa_i-\varphi_i(t))$$

But from (EH) and (26), we also have

$$\sum_{k\in\widehat{I'}\setminus\widehat{I}} e_k(\kappa_k-\varphi_k(t')) < e_\ell\left(\varphi_\ell(t)-\varphi_\ell(t')\right)+e_\ell\sum_{i\in\widehat{I}\setminus\widehat{I'}}(\kappa_i-\varphi_i(t))$$

Then

$$0\geq(\hat{e}-e_\ell)\left(\varphi_\ell(t)-\varphi_\ell(t')\right)+(\hat{e}-e_\ell)\sum_{i\in\widehat{I}\setminus\widehat{I'}}(\kappa_i-\varphi_i(t))$$

which contradicts our assumption.
(i-b) If $\sum_{i\in I}\varphi_i(t)<D(t)$ then, for all $i\in I$, $\varphi_i(t)=\kappa_i$ and (EH) is necessarily false.

We complete the proof of the lemma with the case $I'\neq I$ in Appendix 5.1. □

3.3.2 Towards an equilibrium strategy

In the following we do not explicit a Nash equilibrium. Instead we establish the existence of an interval in which the coupled carbon market Nash equilibria evolve. We demonstrate that there is no possible deviation enabling a Nash equilibrium outside the interval establishing the carbon price. This derives from a series of lemmas in which we explicit the market.

To begin with, we propose an analysis of the three following statements:

Lower price strategy

Consider any strategy $\mathbf{A}^{\mathscr{W}}=(A_1^{\mathscr{W}},\ldots,A_J^{\mathscr{W}})$ such that

$$A_j^{\mathscr{W}}(\tau)=\begin{cases}\mathscr{W}_j(\tau^{\text{lower}}), & \text{for } 0\leq\tau\leq\tau^{\text{lower}}\\ \text{anything admissible}, & \text{for } \tau>\tau^{\text{lower}}.\end{cases} \tag{27}$$

Lemma 3.
(i) $p^{CO_2}(\mathbf{A}^{\mathscr{W}})\geq\tau^{lower}$.
(ii) In the case where $p^{CO_2}(\mathbf{A}^{\mathscr{W}})=\tau^{lower}$*, there is no unilateral favorable deviation that clears the market at a* CO_2 *price lower than* τ^{lower}*.*

Proof. Point *(i)* is a consequence of the definition of $\tau^{\text{lower}} = \sup\{\tau \in [0,\mathfrak{p}],\ \text{s.t. } \mathscr{W}(\tau) > \Omega\}$. Since $A_j^{\mathscr{W}}(\tau) = \mathscr{W}_j(\tau)$ for $\tau \leq \tau^{\text{lower}}$, it follows that $p^{\text{co}_2}(\mathbf{A}^{\mathscr{W}}) = \sup\{\tau \in [0,\mathfrak{p}],\ \text{s.t. } \sum_j A_j^{\mathscr{W}}(\tau) > \Omega\} \geq \tau^{\text{lower}}$.

To prove *(ii)*, first note that, since we assume $p^{\text{co}_2}(\mathbf{A}^{\mathscr{W}}) = \tau^{\text{lower}}$, we have $\varphi_j(\mathbf{A}^{\mathscr{W}}) = \varphi_j(\tau^{\text{lower}}) = \frac{1}{e_j}\mathscr{W}_j(\tau^{\text{lower}})$.

Suppose one producer, say Producer 1, deviates and chooses $\widetilde{A}_1(\cdot)$ instead of $A_1^{\mathscr{W}}(\cdot)$. Suppose the new carbon price $\widetilde{\tau} := p^{\text{co}_2}(\mathbf{A}_{-1}^{\mathscr{W}};\widetilde{A}_1) < \tau^{\text{lower}}$. Since $A_j^{\mathscr{W}}(\widetilde{\tau}^+) = A_j^{\mathscr{W}}(\widetilde{\tau})$ for $j \neq 1$, necessarily we have $\widetilde{A}_1(\widetilde{\tau}^+) < \widetilde{A}_1(\widetilde{\tau})$, by definition of $\widetilde{\tau}$. Then $\Delta(A_1) > 0$ and it follows that $\delta_1(\mathbf{A}_{-1}^{\mathscr{W}};\widetilde{A}_1) = \delta_1(\mathbf{A}^{\mathscr{W}})$.

From the fact that the marginal production costs of all Producers have decreased (the emission cost is $\widetilde{\tau}$ instead of τ^{lower}), it comes that $p^{\text{elec}}(\mathbf{A}_{-1}^{\mathscr{W}};\widetilde{A}_1) \leq p^{\text{elec}}(\mathbf{A}^{\mathscr{W}})$. This means that the part of electricity production capacity that is not covered by allowances (and hence penalized with $\mathfrak{p}$) has a marginal production cost greater than $p^{\text{elec}}(\mathbf{A}_{-1}^{\mathscr{W}};\widetilde{A}_1))$. We then deduce that, at best, $\varphi_1(\mathbf{A}_{-1}^{\mathscr{W}};\widetilde{A}_1) \leq \varphi_1(\mathbf{A}^{\mathscr{W}})$. □

Lemma 4. *Suppose* $\mathbf{A}$ *is such that* $p^{\text{co}_2}(\mathbf{A}) < \tau^{lower}$. *Then* $\mathbf{A}$ *is not a Nash equilibrium.*

Proof. To prove this lemma we exhibit an unilateral deviation from $\mathbf{A}$ of a producer, improving its market share on the electricity market.

• Assume first that at least one producer exists, say Producer 1, such that $\varphi_1(\mathbf{A}) < \kappa_1$ and there exists a tax value $\hat{\tau}_1$ such that $p^{\text{co}_2}(\mathbf{A}) < \hat{\tau}_1 \leq \tau^{\text{lower}}$ and, $\mathscr{W}_1(\tau) = e_1\kappa_1$ for any $\tau \in [p^{\text{co}_2}(\mathbf{A}), \hat{\tau}_1]$.

This means that Producer 1 may sell κ_1, for any tax level τ in $[p^{\text{co}_2}(\mathbf{A}), \hat{\tau}_1]$, and consequently we have $c_1 + \tau e_1 < p^{\text{elec}}(\tau)$ for τ in $[p^{\text{co}_2}(\mathbf{A}), \hat{\tau}_1]$.

Consider a deviation $\widetilde{A}_1$ of player 1, such that the resulting clearing price on CO_2 market, $p^{\text{co}_2}(\mathbf{A}_{-1};\widetilde{A}_1) \in [p^{\text{co}_2}(\mathbf{A}), \hat{\tau}_1]$. From Remark 4, we have

$$p^{\text{elec}}(p^{\text{co}_2}(\mathbf{A}_{-1};\widetilde{A}_1)) \leq p^{\text{elec}}(\mathbf{A}_{-1};\widetilde{A}_1).$$

This means that Producer 1 may sell its overall covered capacity: $\varphi_1(\mathbf{A}_{-1};\widetilde{A}_1) = \frac{1}{e_1}\delta_1(\mathbf{A}_{-1};\widetilde{A}_1)$.

Now we define $\tau \mapsto \widetilde{A}_1(\tau)$ as follows, for $\varepsilon > 0$ arbitrarily small, for $p^{\text{co}_2}(\mathbf{A}) < \tau$,

$$\begin{aligned}
\widetilde{A}_1(\tau) =& e_1\kappa_1 \mathbb{1}_{\{\sum_{j\neq 1} A_j(\tau) + \delta_1(\mathbf{A}) < \Omega\}} \mathbb{1}\{p^{\text{co}_2}(\mathbf{A}) < \tau \leq \hat{\tau}_1\} \\
&+ \left(\Omega - \sum_{j>1} A_j(\tau) - \varepsilon\right) \mathbb{1}_{\{\sum_{j\neq 1} A_j(\tau) + \delta_1(\mathbf{A}) \geq \Omega\}} \mathbb{1}\{p^{\text{co}_2}(\mathbf{A}) \leq \tau \leq \hat{\tau}_1\} \\
&+ A_1(\tau)\mathbb{1}_{\{\tau > \hat{\tau}_1\}} \\
&\text{and} \\
\widetilde{A}_1(p^{\text{co}_2}(\mathbf{A})) =& e_1\kappa_1.
\end{aligned}$$

Note that $\widetilde{A}_1(\tau) \geq A_1(\tau)$ for $p^{\mathrm{CO_2}}(\mathbf{A}) \leq \tau \leq \hat{\tau}_1$, and consequently $p^{\mathrm{CO_2}}(\mathbf{A}_{-1};\widetilde{A}_1) \geq p^{\mathrm{CO_2}}(\mathbf{A})$.

If $p^{\mathrm{CO_2}}(\mathbf{A}_{-1};\widetilde{A}_1) > p^{\mathrm{CO_2}}(\mathbf{A})$, then $e_1\varphi_1(\mathbf{A}_{-1};\widetilde{A}_1)) = \delta(\mathbf{A}_{-1};\widetilde{A}_1) > \delta(\mathbf{A}) \geq e_1\varphi_1(\mathbf{A})$, and we get our favorable deviation.

If $p^{\mathrm{CO_2}}(\mathbf{A}_{-1};\widetilde{A}_1) = p^{\mathrm{CO_2}}(\mathbf{A})$, we observe that when $\Delta(A_1) \geq 0$, we also have $\Delta(\widetilde{A}_1) = 0$. Then by CO_2 market clearing mechanism, Producer 1 get $e_1\kappa_1$ allowances instead of $\delta(\mathbf{A})$ and strictly improves its electricity market share. when $\Delta(A_1) < 0$, we have $\widetilde{A}_1(p^{\mathrm{CO_2}}(\mathbf{A})^+) > A_1(p^{\mathrm{CO_2}}(\mathbf{A})^+)$, that also insure that Producer 1 increase $\delta(\mathbf{A}_{-1};\widetilde{A}_1) > \delta(\mathbf{A})$ (see (10)).

• Assume now that all producers are either such that $\varphi_j(\mathbf{A}) = \kappa_j$ or such that $\varphi_j(\mathbf{A}) < \kappa_j$ and $\mathscr{W}_j(p^{\mathrm{CO_2}}(\mathbf{A})^+) < e_j\kappa_j$. Among the second category, there exists at least one producer (say Producer 1) such that $\varphi_1(\mathbf{A}) < \varphi_1(p^{\mathrm{CO_2}}(\mathbf{A}))$ with $\varphi_1(p^{\mathrm{CO_2}}(\mathbf{A})) > 0$ (unless to contradict $p^{\mathrm{CO_2}}(\mathbf{A}) < \tau^{\mathrm{lower}}$). Here we have used the notation (20) and (21). $\mathscr{W}_1(p^{\mathrm{CO_2}}(\mathbf{A})^+) < e_1\kappa_1$ means that $c_1 + e_1 p^{\mathrm{CO_2}}(\mathbf{A}) = p^{\mathrm{elec}}(p^{\mathrm{CO_2}}(\mathbf{A}))$ (as $p^{\mathrm{elec}}(\cdot)$ is right-continuous).

A strictly favorable deviation $\widetilde{A}_1$ of Producer 1, thus consists in increase its ask at the price $p^{\mathrm{CO_2}}(\mathbf{A})^+$, in order to increase its $\delta((\mathbf{A}_{-1},\widetilde{A}_1))$ (see (10)):

$$\begin{aligned}\widetilde{A}_1(\tau) = &\left(\Omega - \sum_{j>1} A_j(\tau) - \varepsilon\right)\mathbb{1}_{\{p^{\mathrm{CO_2}}(\mathbf{A}) < \tau\}} \\ &+ e_1\kappa_1\mathbb{1}_{\{p^{\mathrm{CO_2}}(\mathbf{A}) = \tau\}}.\end{aligned}$$

Then $p^{\mathrm{CO_2}}((\mathbf{A}_{-1};\widetilde{A}_1)) = p^{\mathrm{CO_2}}(\mathbf{A})$, $\widetilde{A}_1(p^{\mathrm{CO_2}}(\mathbf{A})) \geq A_1(p^{\mathrm{CO_2}}(\mathbf{A}))$, but $\widetilde{A}_1(p^{\mathrm{CO_2}}(\mathbf{A})^+) > A_1(p^{\mathrm{CO_2}}(\mathbf{A})^+)$, for ε sufficiently small. This last inequality guaranties that $\delta_1((\mathbf{A}_{-1};\widetilde{A}_1)) > \delta_1(\mathbf{A})$ and finally $\varphi_1(p^{\mathrm{CO_2}}(\mathbf{A})) \geq \varphi_1((\mathbf{A}_{-1};\widetilde{A}_1)) > \varphi_1(\mathbf{A})$. □

High price strategy

Consider any strategy $\mathbf{A}^{\overline{\mathscr{W}}} = (A_1^{\overline{\mathscr{W}}}, \cdots, A_J^{\overline{\mathscr{W}}})$ such that

$$A_j^{\overline{\mathscr{W}}}(\tau) = \begin{cases} \text{anything admissible}, & \text{for } \tau \leq \tau^{\mathrm{higher}} \\ \overline{\mathscr{W}}_j(\tau), & \text{for } \tau > \tau^{\mathrm{higher}}. \end{cases} \tag{28}$$

Lemma 5.
(i) $p^{CO_2}(\mathbf{A}^{\overline{\mathscr{W}}}) \leq \tau^{higher}$.
(ii) In the case where $p^{CO_2}(\mathbf{A}^{\overline{\mathscr{W}}}) = \tau^{higher}$, *there is no unilateral favorable deviation that clears the market at a* CO_2 *price higher than* τ^{higher}.

Proof. Point *(i)* follows directly from the definition of τ^{higher}.

To prove *(ii)*, suppose one producer, say Producer 1, chooses its strategy $\widetilde{A}_1(\cdot)$ instead of $A_1^{\overline{\mathscr{W}}}(\cdot)$, and that the resulting CO_2 price is $\widetilde{\tau} := p^{\mathrm{CO_2}}(\mathbf{A}_{-1}^{\overline{\mathscr{W}}};\widetilde{A}_1) > \tau^{\mathrm{higher}}$.

Necessarily, due to the definition of $\mathbf{A}^{\overline{\mathscr{W}}}$, this means that $\overline{\mathscr{W}}_1(\widetilde{\tau}) = 0$, which in turn means that $c_1 + \widetilde{\tau} e_1 > p^{\text{elec}}(\widetilde{\tau})$. To conclude, it is sufficient to notice that any Producer $j \neq 1$ obtains what he asks for, i.e. $\delta_j(\mathbf{A}^{\overline{\mathscr{W}}}_{-1};\widetilde{A}_1) = \overline{\mathscr{W}}_j(\widetilde{\tau}^+)$, from which it follows that the *coupled* electricity price equals the *taxed* electricity price: $p^{\text{elec}}(\mathbf{A}^{\overline{\mathscr{W}}}_{-1};\widetilde{A}_1) = p^{\text{elec}}(\widetilde{\tau})$, and then $\varphi_1(\mathbf{A}^{\overline{\mathscr{W}}}_{-1};\widetilde{A}_1) = \overline{\mathscr{W}}_i(\widetilde{\tau}) = 0$ and the deviation of 1 is not favorable. □

Lemma 6. *Suppose* $\mathbf{A}$ *is such that* $p^{co_2}(\mathbf{A}) > \tau^{higher}$*. Then* $\mathbf{A}$ *is not a strong Nash equilibrium.*

Proof. Given $\mathbf{A}$, such that $p^{\text{co}_2}(\mathbf{A}) > \tau^{\text{higher}}$, we consider the coalition of producers $\mathscr{K}$ such that for $j \in \mathscr{K}$, $\delta_j(\mathbf{A}) > 0$ whereas $\overline{\mathscr{W}}_j(p^{\text{co}_2}(\mathbf{A})) = 0$. Consider the following cooperating deviation of $\mathscr{K}$:

$$\widetilde{A}_j(\cdot) = A^{\overline{\mathscr{W}}}_j(\cdot), \quad \text{for } j \in \mathscr{K}.$$

Then $p^{\text{co}_2}(\mathbf{A}_{-\mathscr{K}};\widetilde{A}_{\mathscr{K}}) < p^{\text{co}_2}(\mathbf{A})$ and at least for one member of $\mathscr{K}$, $\delta_j(\mathbf{A}) > 0$ when $\overline{\mathscr{W}}_j(p^{\text{co}_2}(\mathbf{A})) > 0$. This means that $\varphi_j(\mathbf{A}_{-\mathscr{K}};\widetilde{A}_{\mathscr{K}}) > 0$ which is a strictly favorable deviation of j, whereas the situation is unchanged for the others in $\mathscr{K}$ that still produce nothing. We exhibit a coalition that allows a deviation from $\mathbf{A}$ that benefits to all of its members. Then $\mathbf{A}$ is not a strong Nash equilibrium. □

Intermediate strategy

Consider any strategy profile $\mathbf{B} = (B_1, \cdots, B_J)$ satisfying the following:

$$B_j(\tau) = \begin{cases} \overline{\mathscr{W}}_j(\tau), & \text{for } \tau > \tau^{\text{higher}} \\ \text{anything admissible}, & \text{for } \tau^{\text{lower}} < \tau \leq \tau^{\text{higher}} \\ \mathscr{W}_j(\tau^{\text{lower}}), & \text{for } \tau \leq \tau^{\text{lower}}. \end{cases} \tag{29}$$

This is not in general an equilibrium, nevertheless we have the following properties :

Lemma 7.
(i) $p^{co_2}(\mathbf{B}) \in [\tau^{lower}, \tau^{higher}]$.
(ii) If there exists a favorable deviation from a producer, say Producer 1, that chooses $\widetilde{B}_1$ *instead of* B_1*, such that* $p^{co_2}(\mathbf{B}_{-1};\widetilde{B}_1) < \tau^{lower}$*, then there exists another favorable deviation* $\widehat{B}_1$ *such that* $p^{co_2}(\mathbf{B}_{-1};\widehat{B}_1) = \tau^{lower}$*, and such that* $\varphi_1(\mathbf{B}_{-1};\widehat{B}_1) \geq \varphi_1(\mathbf{B}_{-1};\widetilde{B}_1)$.

Proof. Point *(i)* follows directly from Lemma 3-*(i)* and Lemma 5-*(i)*.

To prove point *(ii)*, we denote $\tau_{\mathbf{B}} := p^{\text{co}_2}(\mathbf{B}_{-1};\widetilde{B}_1) < \tau^{\text{lower}}$. We first observe that, as producers $j \neq 1$ are served first on the carbon market,

$$\delta_1(\widetilde{\mathbf{B}}) = \Omega - \sum_{j \neq 1} \mathscr{W}_j(\tau^{\text{lower}}).$$

Define $\widehat{B}_1$, the new deviation of 1 as

$$\widehat{B}_1 = \begin{cases} \widetilde{B}_1(\tau), & \text{for } \tau > \tau^{\text{lower}}, \\ \mathscr{W}_1(\tau^{\text{lower}}), & \text{for } \tau \leq \tau^{\text{lower}}. \end{cases}$$

The CO_2 price is now fixed to $p^{\text{CO}_2}((\mathbf{B}_{-1};\widehat{B}_1)) = \tau^{\text{lower}}$, and from the CO_2 market mechanism it follows that

$$\delta_1((\mathbf{B}_{-1};\widehat{B}_1)) \geq \delta_1((\mathbf{B}_{-1};\widetilde{B}_1)).$$

Since $\widetilde{B}_j(\tau_{\mathbf{B}}) = \widetilde{B}_j(\tau_{\mathbf{B}}^+) = \mathscr{W}_j(\tau^{\text{lower}})$ for any $j \neq 1$, it comes that $\delta_1((\mathbf{B}_{-1};\widetilde{B}_1)) = \Omega - \sum_{j\neq 1} \mathscr{W}_j(\tau^{\text{lower}})$. Indeed, for strategy $(\mathbf{B}_{-1};\widehat{B}_1)$, the producers $j \neq 1$ such that $B_j(\tau^{\text{lower}+}) < \mathscr{W}_j(\tau^{\text{lower}})$ receive a quantity of quotas $\delta_j((\mathbf{B}_{-1};\widehat{B}_1) \leq \mathscr{W}_j(\tau^{\text{lower}})$, from which it follows that $\delta_1((\mathbf{B}_{-1};\widehat{B}_1)) = \Omega - \sum_j \delta_j((\mathbf{B}_{-1};\widehat{B}_1)) \geq \delta_1((\mathbf{B}_{-1};\widehat{B}_1))$. We also deduce that $\varphi_1((\mathbf{B}_{-1};\widehat{B}_1)) = \frac{1}{e_1}\delta_1((\mathbf{B}_{-1};\widehat{B}_1))$.

To conclude, it is sufficient to notice that $\varphi_1((\mathbf{B}_{-1};\widehat{B}_1)) = \frac{1}{e_1}\delta_1((\mathbf{B}_{-1};\widehat{B}_1)) \geq \frac{1}{e_1}\delta_1((\mathbf{B}_{-1};\widetilde{B}_1)) \geq \varphi_1((\mathbf{B}_{-1};\widetilde{B}_1))$. □

The following corollary is a direct consequence of Lemmas 4, 5, 6 and 7.

Corollary 1. *Let* $\mathbf{E}$ *be a (strong) Nash equilibrium. Then the following* $\mathbf{E}'$ *is also a (strong) Nash equilibrium:*

$$E'_j(\tau) = \begin{cases} \overline{\mathscr{W}}_j(\tau), & \textit{for } \tau > \tau^{higher} \\ E_j(\tau), & \textit{for } \tau^{lower} < \tau \leq \tau^{higher} \\ \mathscr{W}_j(\tau^{lower}), & \textit{for } \tau \leq \tau^{lower} \end{cases} \tag{30}$$

It is worthy of mentioning that same results of the section 3.3 apply when producers have an electricity production power plants portfolio, or when one modify the maximal amount $\mathscr{E}_j$ of allowances to buy. The interval remain relevant, with straightforward modification on the functions $\mathscr{W}_j(\cdot)$ and $\overline{\mathscr{W}}_j(\cdot)$ and the related price bounds τ^{lower} and τ^{higher}.

4 Conclusion

Once CO_2 is emitted into the atmosphere, it remains there for more than a century. Estimating its value is an essential indicator for efficiently defining policy. Carbon valuation is crucial for designing markets that foster emission reductions. In this paper, we established the links between an electricity market and a carbon auction market through an analysis of electricity producers' strategies. We proved that they lead to the interval where relevant Nash equilibria evolve, enabling the computation of equilibrium prices on both markets. It has been established that Nash equilibria driver on the carbon market rely more on the producers' emission rate than on their marginal costs. For each producer, each equilibrium derives the level of electricity produced and the CO_2 emissions covered.

For a given design and set of players, the information provided by the interval may be interpreted as a diagnosis of market behavior in terms of prices and volume.

In addition to this analysis of the Nash equilibrium we plan to analyze the electricity production mix, with a particular focus on renewable shares that do not participate in emissions.

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5 Appendix

5.1 End of the proof of Lemma 2

(ii) We examine the situation $I(t') \neq I(t)$

To shorten the expressions, we still adopt the following shorten notation

$$I(t) = I \quad \text{and} \quad I(t') = I'.$$

$$I(t) \cap I'(t) = II'$$

We break down I and I' into the sets II', $I \backslash I'$ and $I' \backslash I$. We denote by $\widehat{I}$ the set of index $i \in I$ such that $c_i + te_i = \underline{p}(t)$. In particular, when $j \in I \backslash \widehat{I}$, then $\varphi_j(t) = \kappa_j$.

We first derive some generic relations between the emission rates for these sets.

Among the indexes in the set II', we observe that at most one index exists (say ℓ) in the set $\widehat{I} \cap \widehat{I'}$. If $j \in \widehat{I} \backslash \widehat{I'}$, if $k \in \widehat{I'} \backslash \widehat{I}$, then, by the definition of the sets

$$\begin{aligned}
c_j + e_j t &= c_\ell + e_\ell t \\
c_j + e_j t' &< c_\ell + e_\ell t' \\
c_k + e_k t &< c_j + e_j t \\
c_k + e_k t' &= c_\ell + e_\ell t' \\
c_k + e_k t &< c_\ell + e_\ell t \\
c_j + e_j t' &< c_k + e_k t
\end{aligned}$$

from which, we easily deduce that

$$\hat{e} := \max\left\{e_j, j \in II' \cap \left(\widehat{I} \backslash \widehat{I'}\right)\right\} < e_\ell < \min\left\{e_k, k \in II' \cap \left(\widehat{I'} \backslash \widehat{I}\right)\right\} := \hat{e}'. \qquad (31)$$

For $j \in I \backslash I'$ and $k \in I' \backslash I$, we have

$$c_j + e_j t < c_k + e_k t$$
$$c_j + e_j t' > c_k + e_k t'$$

from which, we also easily deduce that

$$\max\{e_k, k \in I' \backslash I\} < \min\{e_j, j \in I \backslash I'\}. \tag{32}$$

For the same j and k, for $(\hat{c}, \hat{e})$ representative of index in $I\!I' \cap \widehat{I} \setminus \widehat{I'}$, and $(\hat{c}', \hat{e}')$ representative of index in $I\!I' \cap \widehat{I'} \setminus \widehat{I}$, we also have

$$\begin{array}{ll} c_j + e_j t & \le \hat{c} + \hat{e} t \\ c_j + e_j t' & > \hat{c} + \hat{e} t' \end{array} \quad \text{and} \quad \begin{array}{ll} c_k + e_k t & > \hat{c}' + \hat{e}' t \\ c_k + e_k t' & \le \hat{c}' + \hat{e}' t' \end{array}$$

from which, we deduce that

$$\begin{aligned} \min\{e_j, j \in I \backslash I'\} &> (e_\ell, \hat{e}) \vee \max\{e_k, k \in I' \backslash I\} \\ \max\{e_k, k \in I' \backslash I\} &< (e_\ell, \hat{e}') \wedge \min\{e_j, j \in I \backslash I'\}. \end{aligned} \tag{33}$$

We divide your analysis in cases. In the first one the demand is fully satisfied for the price $p^{\text{elec}}(t)$.

(ii-a) If $\sum_{i \in I} \varphi_I(t) = D(p^{\text{elec}}(t))$

$$\sum_{i \in I \backslash I'} \varphi_I + \sum_{i \in I\!I'} \varphi_i(t) = D(p^{\text{elec}}(t)) \ge D(p^{\text{elec}}(t')) \ge \sum_{i \in I\!I'} \varphi_i(t') + \sum_{i \in I' \backslash I} \varphi_i(t') \quad \text{(DC)}$$

$$\sum_{i \in I \backslash I'} \varphi_i(t) e_i + \sum_{i \in I\!I'} \varphi_i(t) e_i < \sum_{i \in I\!I'} \varphi_i(t') e_i + \sum_{i \in I' \backslash I} \varphi_i(t) e_i \quad \text{(EH)}$$

We must then examine the following two subcases, relative to the situations where the demand is satisfied or not at the price $p^{\text{elec}}(t')$.

(ii-a-1) If $\sum_{i \in I'} \varphi_i(t') < D(p^{\text{elec}}(t'))$, then $\varphi_i(t') = \kappa_i$ for all $i \in I'$ and

$$\sum_{j \in I \backslash I'} \varphi_j(t) + \sum_{i \in I\!I'} \varphi_i(t) > \sum_{i \in I\!I'} \kappa_i + \sum_{k \in I' \backslash I} \kappa_k \quad \text{(DC)}$$

$$\sum_{j \in I \backslash I'} \varphi_j(t) e_j + \sum_{i \in I\!I'} \varphi_i(t) e_i < \sum_{i \in I\!I'} \kappa_i e_i + \sum_{k \in I' \backslash I} \kappa_k e_k \quad \text{(EH)}$$

As $\varphi_i(t) = \kappa_i$ when $i \in (I \backslash \widehat{I}) \cap I\!I'$, we can simplify the two sides of (DC) and (EH) by the sum over $(I \backslash \widehat{I}) \cap I\!I'$. The remaining part of $I\!I'$ is $\{\ell\} \cup \left(\widehat{I} \backslash \widehat{I'} \cap I\!I'\right)$:

$$\sum_{j \in I \backslash I'} \varphi_j(t) + \varphi_\ell + \sum_{i \in \widehat{I} \backslash \widehat{I'} \cap I\!I'} \varphi_i(t) > \kappa_\ell + \sum_{i \in \widehat{I} \backslash \widehat{I'} \cap I\!I'} \kappa_i + \sum_{k \in I' \backslash I} \kappa_k \quad \text{(DC)}$$

$$\sum_{j \in I \backslash I'} e_j \varphi_j(t) + e_\ell \varphi_\ell + \sum_{i \in \widehat{I} \backslash \widehat{I'} \cap I\!I'} e_i \varphi_i(t) < e_\ell \kappa_\ell + \sum_{i \in \widehat{I} \backslash \widehat{I'} \cap I\!I'} e_i \kappa_i + \sum_{k \in I' \backslash I} e_k \kappa_k \quad \text{(EH)}$$

Then, we multiply (DC) by $\bar{e} := (e_\ell, \hat{e}) \vee \max\{e_k, k \in I' \backslash I\}$, and we obtain by (33)

$$\sum_{j\in I\setminus I'} e_j\varphi_j(t)+\bar{e}\varphi_\ell+\bar{e}\sum_{i\in\widehat{I}\setminus\widehat{I'}\cap II'}\varphi_i(t)>\bar{e}\kappa_\ell+\bar{e}\sum_{i\in\widehat{I}\setminus\widehat{I'}\cap II'}\kappa_i+\sum_{k\in I'\setminus I} e_k\kappa_k.$$

We subtract with (EH) :

$$(\bar{e}-e_\ell)\varphi_\ell+\sum_{i\in\widehat{I}\setminus\widehat{I'}\cap II'}(\bar{e}-e_i)\varphi_i(t)>(\bar{e}-e_\ell)\kappa_\ell+\sum_{i\in\widehat{I}\setminus\widehat{I'}\cap II'}(\bar{e}-e_i)\kappa_i.$$

But $\bar{e}\geq e_\ell$ when ℓ exists, and $\bar{e}\geq\hat{e}\geq e_i$ for $i\in\widehat{I}\setminus\widehat{I'}\cap II'$. So we obtain our contradiction.

(ii-a-2) If $\sum_{i\in I'}\varphi_i(t')=D(p^{\text{elec}}(t'))$, then

$$\sum_{j\in I\setminus I'}\varphi_j(t)+\sum_{i\in II'}\varphi_i(t)>\sum_{i\in II'}\varphi_i(t')+\sum_{k\in I'\setminus I}\varphi_k(t') \quad \text{(DC)}$$

$$\sum_{j\in I\setminus I'}\varphi_j(t)e_j+\sum_{i\in II'}\varphi_i(t)e_i<\sum_{i\in II'}\varphi_i(t')e_i+\sum_{k\in I'\setminus I}\varphi_k(t')e_k. \quad \text{(EH)}$$

We decompose $I\setminus I'=\left(I\setminus(I'\cup\widehat{I})\right)\cup\widehat{I}\setminus I'$ and $I'\setminus I=\left(I'\setminus(I\cup\widehat{I'})\right)\cup\widehat{I'}\setminus I$:

$$\begin{aligned}
&\sum_{j\in I\setminus(I'\cup\widehat{I})}\kappa_j+\sum_{j\in\widehat{I}\setminus I'}\varphi_j(t)+\sum_{i\in II'}\varphi_i(t)\\
&>\sum_{i\in II'}\varphi_i(t')+\sum_{k\in\widehat{I'}\setminus I}\varphi_k(t')+\sum_{k\in I'\setminus(I\cup\widehat{I'})}\kappa_k \quad \text{(DC)}\\
&\sum_{j\in I\setminus(I'\cup\widehat{I})}e_j\kappa_j+\sum_{j\in\widehat{I}\setminus I'}e_j\varphi_j(t)+\sum_{i\in II'}e_i\varphi_i(t)\\
&<\sum_{i\in II'}e_i\varphi_i(t')+\sum_{k\in\widehat{I'}\setminus I}e_k\varphi_k(t')+\sum_{k\in I'\setminus(I\cup\widehat{I'})}e_k\kappa_k. \quad \text{(EH)}
\end{aligned}$$

We also break down the set $II'=(I\cap I')$:

$$II'=\left(II'\cap\{\ell\}\right)\cup\left(II'\cap\widehat{I}\setminus\widehat{I'}\right)\cup\left(II'\cap\widehat{I'}\setminus\widehat{I}\right)\cup\left(I\setminus\widehat{I}\cap I'\setminus\widehat{I'}\right).$$

$$\begin{aligned}
&\sum_{j\in I\setminus(I'\cup\widehat{I}))}\kappa_j+\sum_{j\in\widehat{I}\setminus I'}\varphi_j(t)+\varphi_\ell(t)+\sum_{i\in\widehat{I}\setminus\widehat{I'}\cap II'}\varphi_i(t)+\sum_{i\in\widehat{I'}\setminus\widehat{I}\cap II'}\varphi_i(t)\\
&>\varphi_\ell(t')+\sum_{i\in\widehat{I}\setminus\widehat{I'}\cap II'}\varphi_i(t')+\sum_{i\in\widehat{I'}\setminus\widehat{I}\cap II'}\varphi_i(t')+\sum_{k\in\widehat{I'}\setminus I}\varphi_k(t')+\sum_{k\in I'\setminus(I\cup\widehat{I'})}\kappa_k \quad \text{(DC)}\\
&\sum_{j\in I\setminus(I'\cup\widehat{I})}e_j\kappa_j+\sum_{j\in\widehat{I}\setminus I'}e_j\varphi_j(t)+e_\ell\varphi_\ell(t)+\sum_{i\in\widehat{I}\setminus\widehat{I'}\cap II'}e_i\varphi_i(t)+\sum_{i\in\widehat{I'}\setminus\widehat{I}\cap II'}e_i\varphi_i(t)\\
&<e_\ell\varphi_\ell(t')+\sum_{i\in\widehat{I}\setminus\widehat{I'}\cap II'}e_i\varphi_i(t')+\sum_{i\in\widehat{I'}\setminus\widehat{I}\cap II'}e_i\varphi_i(t')+\sum_{k\in\widehat{I'}\setminus I}e_k\varphi_k(t')+\sum_{k\in I'\setminus(I\cup\widehat{I'})}e_k\kappa_k \quad \text{(EH)}
\end{aligned}$$

For index i in the last subset $(I\backslash\widehat{I}\cap I'\backslash\widehat{I'})$, we have $\varphi_i(t)=\kappa_i$ and $\varphi_i(t')=\kappa_i$, so we simplify (DC) and (EH) from this last subset. Thus,

$$\begin{aligned}
&\sum_{j\in I\backslash(I'\cup\widehat{I})}\kappa_j+\sum_{j\in\widehat{I}\backslash I'}\varphi_j(t)+\varphi_\ell(t)+\sum_{i\in\widehat{I}\backslash\widehat{I'}\cap II'}\varphi_i(t)+\sum_{i\in\widehat{I'}\backslash\widehat{I}\cap II'}\kappa_i\\
&>\varphi_\ell(t')+\sum_{i\in\widehat{I}\backslash\widehat{I'}\cap II'}\kappa_i+\sum_{i\in\widehat{I'}\backslash\widehat{I}\cap II'}\varphi_i(t')+\sum_{k\in\widehat{I'}\backslash I}\varphi_k(t')+\sum_{k\in I'\backslash(I\cup\widehat{I'})}\kappa_k \quad \text{(DC)}\\
&\sum_{j\in I\backslash(I'\cup\widehat{I})}e_j\kappa_j+\sum_{j\in\widehat{I}\backslash I'}e_j\varphi_j(t)+e_\ell\varphi_\ell(t)+\sum_{i\in\widehat{I}\backslash\widehat{I'}\cap II'}e_i\varphi_i(t)+\sum_{i\in\widehat{I'}\backslash\widehat{I}\cap II'}e_i\kappa_i\\
&<e_\ell\varphi_\ell(t')+\sum_{i\in\widehat{I}\backslash\widehat{I'}\cap II'}e_i\kappa_i+\sum_{i\in\widehat{I'}\backslash\widehat{I}\cap II'}e_i\varphi_i(t')+\sum_{k\in\widehat{I'}\backslash I}e_k\varphi_k(t')+\sum_{k\in I'\backslash(I\cup\widehat{I'}}e_k\kappa_k \quad \text{(EH)}
\end{aligned}$$

We multiply (DC) by $\bar{e}:=(e_\ell,\hat{e})\vee\max\{e_k,k\in I'\backslash I\}$, we get by (33)

$$\begin{aligned}
&\sum_{j\in I\backslash(I'\cup\widehat{I})}e_j\kappa_j+\sum_{j\in\widehat{I}\backslash I'}e_j\varphi_j(t)+\bar{e}\varphi_\ell(t)+\bar{e}\sum_{i\in\widehat{I}\backslash\widehat{I'}\cap II'}\varphi_i(t)+\bar{e}\sum_{i\in\widehat{I'}\backslash\widehat{I}\cap II'}\kappa_i\\
&>\bar{e}\varphi_\ell(t')+\bar{e}\sum_{i\in\widehat{I}\backslash\widehat{I'}\cap II'}\kappa_i+\bar{e}\sum_{i\in\widehat{I'}\backslash\widehat{I}\cap II'}\varphi_i(t')+\sum_{k\in\widehat{I'}\backslash I}e_k\varphi_k(t')+\sum_{k\in I'\backslash(I\cup\widehat{I'}}e_k\kappa_k
\end{aligned}$$

We subtract (EH)

$$\begin{aligned}
&(\bar{e}-e_\ell)\varphi_\ell(t)+\sum_{i\in\widehat{I}\backslash\widehat{I'}\cap II'}(\bar{e}-e_i)\varphi_i(t)+\sum_{i\in\widehat{I'}\backslash\widehat{I}\cap II'}(\bar{e}-e_i)\kappa_i\\
&>(\bar{e}-e_\ell)\varphi_\ell(t')+\sum_{i\in\widehat{I}\backslash\widehat{I'}\cap II'}(\bar{e}-e_i)\kappa_i+\sum_{i\in\widehat{I'}\backslash\widehat{I}\cap II'}(\bar{e}-e_i)\varphi_i(t')
\end{aligned}$$

We arrange the terms

$$\begin{aligned}
&(\bar{e}-e_\ell)\varphi_\ell(t)+\sum_{i\in\widehat{I}\backslash\widehat{I'}\cap II'}(\bar{e}-e_i)\varphi_i(t)+\sum_{i\in\widehat{I'}\backslash\widehat{I}\cap II'}(\bar{e}-e_i)\kappa_i\\
&>(\bar{e}-e_\ell)\varphi_\ell(t')+\sum_{i\in\widehat{I}\backslash\widehat{I'}\cap II'}(\bar{e}-e_i)\kappa_i+\sum_{i\in\widehat{I'}\backslash\widehat{I}\cap II'}(\bar{e}-e_i)\varphi_i(t')
\end{aligned}$$

If ℓ exists, then $\bar{e}=e_\ell$ and

$$\begin{aligned}
\sum_{i\in\widehat{I'}\backslash\widehat{I}\cap II'}(e_\ell-e_i)\left(\kappa_i-\varphi_i(t')\right)&>\sum_{i\in\widehat{I}\backslash\widehat{I'}\cap II'}(e_\ell-e_i)\left(\kappa_i-\varphi_i(t)\right)\\
\sum_{i\in\widehat{I'}\backslash\widehat{I}\cap II'}(e_\ell-\hat{e}')\left(\kappa_i-\varphi_i(t')\right)&>\sum_{i\in\widehat{I}\backslash\widehat{I'}\cap II'}(e_\ell-\hat{e})\left(\kappa_i-\varphi_i(t)\right).
\end{aligned} \tag{34}$$

But $\hat{e}<e_\ell<\hat{e}'$, and the contradiction follows.

If ℓ does not exist, then $\bar{e}=\hat{e}\vee\max\{e_k,k\in I'\backslash I\}$

$$\begin{aligned}\sum_{i\in\widehat{I'}\setminus\widehat{I}\cap II'}(\bar{e}-e_i)\left(\kappa_i-\varphi_i(t')\right) &> \sum_{i\in\widehat{I}\setminus\widehat{I'}\cap II'}(\bar{e}-e_i)\left(\kappa_i-\varphi_i(t)\right)\\ \sum_{i\in\widehat{I'}\setminus\widehat{I}\cap II'}(\bar{e}-\hat{e}')\left(\kappa_i-\varphi_i(t')\right) &> \sum_{i\in\widehat{I}\setminus\widehat{I'}\cap II'}(\bar{e}-\hat{e})\left(\kappa_i-\varphi_i(t)\right)\end{aligned} \tag{35}$$

But $\max\{e_k, k\in I'\setminus I\} < \hat{e}'$, and the contradiction follows.

(ii-b) If $\sum_{i\in I}\varphi_i(t) < D(p^{\text{elec}}(t))$ then for all $i\in I$, $\varphi_i(t)=\kappa_i$.
(ii-b 1) If $\sum_{i\in I'}\varphi_i(t') < D(p^{\text{elec}}(t'))$, then $\varphi_i(t')=\kappa_i$ for all $i\in I'$. Moreover, we have that $\mathscr{O}(t,\underline{p}(t)) \geq D(\underline{p}(t))+\varepsilon) \geq D(\underline{p}(t')) > \mathscr{O}(t',\underline{p}(t'))$ and (DC)-(EH) becomes

$$\sum_{j\in I\setminus I'}\kappa_j > \sum_{k\in I'\setminus I}\kappa_k \quad \text{(DC)}$$

$$\sum_{j\in I\setminus I'}e_j\kappa_j < \sum_{k\in I'\setminus I}e_k\kappa_k \quad \text{(EH)}$$

Then, we multiply (DC) by $\min\{e_j; j\in I\setminus I'\} \geq \max\{e_k; k\in I'\setminus I\}$, and we obtain a contradiction with (EH).

(ii-b-2) If $\sum_{i\in I'}\varphi_i(t') = D(p^{\text{elec}}(t'))$, we go back to the analysis of the case **(ii-a-2)**, with the main difference that all quantities $\varphi_i(t)$ are now equal to κ_i. We go to inequalities (34) and (35) which are simplified as the right-had sides are now zero. The contradiction follows with the same arguments.

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